Kinematics Modeling and Control of Spherical Rolling Contact Joint and Manipulator

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***Abstract*—Rolling contact joints are attracting increasing interest in applications to robotic fingers and manipulators, due to the potential of the absence of abrasion wear, simplification of the controller, enlargement of reachable configurations, etc. This paper first proposes a novel 2-DOF spherical rolling contact (*SRC*) joint, with the joint model elements being formulated, including such as *rotation matrix*, *position vector*, *free modes*, as with those of classic joints. As an application, a new kind of serial manipulator formed by the *SRC* joints is presented, and its forward and inverse kinematics are modeled. The motions of the 2-DOF *SRC* joint and manipulator are implemented using the FreeBOT, and the control method is proposed for the FreeBOT, such that the *SRC* joint and manipulator realize the motion control. The kinematics and control of the 2-DOF *SRC* joint and manipulator are validated using physics simulations and on a real manipulator formed by FreeBOTs.**

***Index Terms*—rolling contact joint, manipulator, modular self-reconfigurable robot, forward kinematics, inverse kinematics**

NOMENCLATURE

B_i	links of the manipulator
$\boldsymbol{J}_{B_n}$	Jacobian matrix of the *SRC* joint manipulator
$\boldsymbol{K}_p$	end-effector position control gain
$\boldsymbol{K}_a$	end-effector orientation control gain
k_{φ_i}	*steering* control gain of the driving trolley i
k_{s_i}	*forward* control gain of the driving trolley i
l_i	radius of the sphere body B_i, m
$l_{i,w}$	wheel interval of the driving trolley i, m
n	number of the joints
P_i	virtual tangent plane between B_i and B_{i-1}
$\boldsymbol{q}_i$	unit quaternioin that represents the orientation of B_i with respect to B_{i-1}
$\boldsymbol{q}_{i,1}$	unit quaternioin that represents the orientation of B_i with respect to P_i
$\boldsymbol{q}_{i-1,2}$	unit quaternioin that represents the orientation of B_{i-1} with respect to P_i
${}^{*}\boldsymbol{R}_{\#}$	rotation matrix of body $\#$ relative to body $*$
${}^{*}_{**}\boldsymbol{r}_{\#}$	position of body $\#$ with respect to $*$ and expressed in the frame $**$, m
$r_{i,w}$	wheel radius of the driving trolley i, m
$r_{i,out}$	outer radius of the FreeBOT sheel, m
$r_{i,in}$	inner radius of the FreeBOT sheel, m
r	right-superscript that represents reference values
s_i	*forward* arc distance of the driving trolley i, m
u_{if}	*forward* speed of the driving trolley i, m/s
u_{is}	*steering* speed of the driving trolley i, rad/s
w_{ir}	rotation speed of the right wheel, rad/s
w_{il}	rotation speed of the left wheel, rad/s
$\boldsymbol{\Phi}_i$	free modes matrix of the *SRC* joint i
${}^{*}_{**}\boldsymbol{v}_{\#}$	linear velocity of body $\#$ with respect to $*$ and expressed in the frame $**$, m/s
${}^{*}_{**}\boldsymbol{\omega}_{\#}$	angular velocity of body $\#$ with respect to $*$ and expressed in the frame $**$, rad/s
$\boldsymbol{\omega}$	*SRC* joint velocity set of the manipulator, rad/s
φ_i	*steering* angle of the driving trolley i, rad
η_i	scalar part of the unit quaternion $\boldsymbol{q}_i$
$\boldsymbol{\epsilon}_i,$	vector part of the unit quaternion $\boldsymbol{q}_i$
$\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}$	B_i rotates through the angle $\Delta\theta_{i,1}$ around the unit axis $\hat{\boldsymbol{e}}_{i,1}$ relative to P_i
$\Delta\theta_{i-1,2}\hat{\boldsymbol{e}}_{i-1,2}$	B_{i-1} rotates through the angle $\Delta\theta_{i-1,2}$ around the unit axis $\hat{\boldsymbol{e}}_{i-1,2}$ relative to P_i
$\times$	cross product of two vectors
$\tilde{\square}$	conjugate of a quaternion

This work was supported by the project 62073274 from the National Natural Science Foundation of China and the funding AC01202101103 from the Shenzhen Institute of Artificial Intelligence and Robotics for Society. (*Corresponding author: Tin Lun Lam.*)

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I. INTRODUCTION

ROLLING contact in which one body rolls without slipping over the surface of the other is widespread in robotics, such as spherical robots [1], multifingered robotic hands [2, 3], and manipulator joints [4, 5]. When forming a rolling contact (*RC*) joint, it has the advantages of the absence of abrasion wear, simplification of the controller design, and enlargement of reachable configurations due to the nature of non-holonomic constraints [3, 6]. Research on single joint having multi-degree of freedom (DOF) that can be actuated has been on going, which will make the robots smaller, less costly, or more functional than conventional robots [7]. However, few works have proposed multi-DOF *RC* joints, nor have the joint modeling, implementation, and applications been studied.

Rolling contact actually encompasses several different geometries, sphere on a plane or on another sphere, for example. Hence, rolling contact pair allows relative motions with different dimensions, forming rolling contact kinematic joint with one, two, or three DOFs [8]. Regardless of the number of DOFs, the no-slip condition associated with rolling contact, as non-holonomic constraints [8], represents the principle basis of its kinematics modeling, which requires the instantaneous relative velocity between the contact points on the two bodies be zero. Li et al. in [6] review the geometries of contact surfaces and the kinematics of contact. Using Darboux frame, the kinematics of spin-rolling contact motion are studied in [3, 9, 10]. These works provide alternative ways of describing rolling contact motion in a general sense, which are inspirational for formulating *RC* joints model.

Rolling contact joints are attracting increasing attention on their potential applications to robotic fingers [11–16], manipulator [4, 5, 17–22], etc. The kinematics of the 1-DOF circular rolling contact (*CRC*) joint is modeled in [11], which is used to form a 2-joint robot finger. To improve compliance of the robotic hands, *RC* joints have been adopted to manufacture sophisticated fingers [12–14]. The optimal design of underactuated fingers and hand implant using *RC* joint is studied to increase performance in robotic and prosthetic hands applications [15, 16]. A 3-DOF serial chain [17] and a 3-DOF planar parallel mechanism [4] using *CRC* joints is designed, respectively, with the forward and inverse kinematics being analyzed. Contributing to the compactness, miniaturization potential, and lower part count, the *RC* joints are used to construct hyper-redundant manipulators for minimally invasive surgery [5, 19, 20], with the performance of the *RC* joints in terms of payload, force, and stiffness are continuously being optimized [18, 21, 22]. It is worth noting that so far only 1-DOF *CRC* joint has been explored, and the robotic fingers and manipulators perform only planar motions. To the authors' knowledge, there is only one work, inspired by human knee anatomy, considering using coupled spheres rather than cylinders to form a *RC* joint for usage as a knee joint in lower limb exoskeletons [23]. However, the formed *RC* joint is not specifically studied, whose kinematics are equivalently simulated using those of multiple combined revolute joints. A 1-DOF rolling contact pair that can generate spatial relative motion between links is proposed in [24–26], however, the allowed spatial rolling motion needs be pre-specified, then the shapes of the both pairing elements that can generate the specified rolling motions must be manufactured.

This paper first proposes a 2-DOF spherical rolling contact joint, abbreviated as *SRC* joint. The 2-DOF *SRC* joint is formed by relative rolling motions about two axes on the common tangent plane of the two sphere shells that may have different radii. The joint model of the 2-DOF *SRC* joint is formulated in this paper, which includes *rotation matrix*, *position vector*, *free modes*, etc., as with those summarized for classic joints, revolute joint for example [8]. Using the proposed 2-DOF *SRC* joints, a new kind of serial manipulator is presented, whose forward and inverse kinematics modeling are performed using the 2-DOF *SRC* joint model. Due to the geometric symmetry of the spherical units, the *SRC* joint manipulator has the advantage of performing tasks in environments with obstacles or narrow spaces.

The FreeBOT is used to implement the motions of the 2-DOF *SRC* joint and manipulator [27], where a driving trolley moves forward and steers inside the spherical iron shell to drive the joint motions, and a magnet at trolley's bottom attracts the other body not to separate. This paper also develops

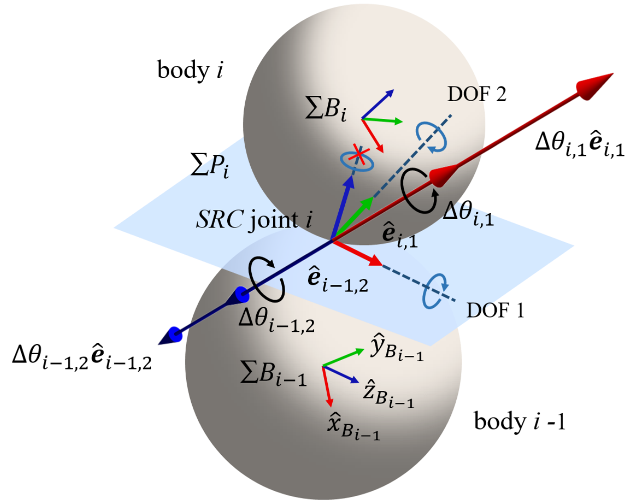


Fig. 1: Notations of the 2-DOF *SRC* joint and its Angle-axis orientation representation

a physics simulation system for the proposed *SRC* joint and manipulator, and builds a real *SRC* joint manipulator using FreeBOTs, such that the 2-DOF *SRC* joint model, manipulator kinematics and control method can be validated.

The main contributions of this paper are threefold:

(1) a new kind of robotic joint, termed as 2-DOF *SRC* joint, is proposed, the joint model is formulated as with its other classic joint counterparts;

(2) a new kind of serial manipulator composed of the *SRC* joints is presented, whose forward and inverse kinematics are modeled;

(3) the motion control of the 2-DOF *SRC* joint and manipulator is realized using FreeBOT, which provides an alternative way for actively actuating the 2-DOF *SRC* joint.

The rest of the paper is organized in the following way. Section II formulates the 2-DOF *SRC* joint model. Section III derives the forward and inverse kinematics model of the *SRC* joint manipulator. Section IV uses the FreeBOT to implement the motion control of the 2-DOF *SRC* joint and manipulator. Section V validates the proposed 2-DOF *SRC* joint model and the performance of the *SRC* joint manipulator with physics simulation and physical experiments, respectively. Section VI concludes the paper.

II. Spherical rolling contact joint

A *RC* joint is formed when one body *rolls without slipping* over the surface of the other [8]. As shown in Fig. 1, this paper introduces a novel 2-DOF *SRC* joint, including its *rotation matrix*, *position vector*, *free modes*, etc, which connects two spherical bodies that may have different geometric sizes.

The *SRC* joint provides two key constraints that combine to form a 2-DOF joint. The first constraint ensures that the bodies do not separate at the point of contact. The second constraint is *rolling without slipping* about two axes on the common tangent plane between the two bodies.

As shown in Fig. 1, this paper represents the bodies through their body-fixed frames $\sum B_i$ and $\sum B_{i-1}$ located at the sphere centers, with their radii denoted as l_i and $l_{i\text{-}1}$, respectively, that $l_i \neq l_{i\text{-}1}$ is possible. For ease of joint modeling derivation, we introduce the virtual common tangent plane P_i between the bodies, and its body-fixed frame $\sum P_i$ located at the point of contact of the two bodies. Supposing the 2-DOF *SRC* joint axes are aligned with the $\hat{x}$ and $\hat{y}$ axes of the frame $\sum P_i$, the initial joint state corresponds to that $\sum B_i$, $\sum B_{i\text{-}1}$ and $\sum P_i$ have the same orientation.

A. Rotation Matrix

The *rotation matrix* of the *SRC* joint transforms a vector expressed in frame $\sum B_i$ to a vector expressed in frame $\sum B_{i\text{-}1}$, which represents the orientation of frame $\sum B_i$ relative to frame $\sum B_{i\text{-}1}$. There are different ways for the representation of the orientation [8], such as Euler angles, Angle-axis, and Quaternions, which can be used to equivalently calculate the rotation matrix.

This paper uses unit quaternion to describe the orientation of the body B_i with respect to the body $B_{i\text{-}1}$, denoted as $\boldsymbol{q}_i = [\eta_i, \boldsymbol{\epsilon}_i]^\top \in \mathbb{R}^4$, with η_i and $\boldsymbol{\epsilon}_i$ being the scalar and vector parts of the quaternion, respectively.

Defining the *SRC* joint velocity ${}^{B_{i\text{-}1}}_{P_i}\boldsymbol{\omega}_{B_i}$, which represents the angular velocity of $\sum B_i$ with respect to $\sum B_{i\text{-}1}$ and expressed in the frame $\sum P_i$ (note that the *SRC* joint axes are aligned with the axes of the frame $\sum P_i$), how the *SRC* joint velocity ${}^{B_{i\text{-}1}}_{P_i}\boldsymbol{\omega}_{B_i}$ makes the rotation matrix ${}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i}$ change is the focus of this subsection. A more physically meaningful set of angular velocities ${}^{P_i}_{P_i}\boldsymbol{\omega}_{B_i}$ that closely relate to the driving unit of the *SRC* joint, as shown in Section IV-A, is defined, it has the following relationship with ${}^{B_{i\text{-}1}}_{P_i}\boldsymbol{\omega}_{B_i}$:

$$ {}^{B_{i\text{-}1}}_{P_i}\boldsymbol{\omega}_{B_i} = \frac{l_i + l_{i\text{-}1}}{l_i}\, {}^{P_i}_{P_i}\boldsymbol{\omega}_{B_i}. \tag{1} $$

Defining the unit quaternion $\boldsymbol{q}_{i,1} = [\eta_{i,1}, \boldsymbol{\epsilon}_{i,1}]^\top \in \mathbb{R}^4$ to represent the orientation of the body B_i with respect to the tangent plane P_i, its derivative can be calculated from ${}^{P_i}_{P_i}\boldsymbol{\omega}_{B_i}$ as

$$ \begin{cases} \dot{\eta}_{i,1} = -\dfrac{1}{2}\, {}^{P_i}_{P_i}\boldsymbol{\omega}_{B_i}^\top \boldsymbol{\epsilon}_{i,1}, \\ \dot{\boldsymbol{\epsilon}}_{i,1} = \dfrac{1}{2}\left(\eta_{i,1}\, {}^{P_i}_{P_i}\boldsymbol{\omega}_{B_i} - \boldsymbol{\epsilon}_{i,1} \times {}^{P_i}_{P_i}\boldsymbol{\omega}_{B_i}\right), \end{cases} \tag{2} $$

where '$\times$' represents the cross product.

The unit quaternion $\boldsymbol{q}_{i,1}$ is obtained by integrating over Eq. (2). In order to calculate $\boldsymbol{q}_i$ from $\boldsymbol{q}_{i,1}$, we define the unit quaternion $\boldsymbol{q}_{i\text{-}1,2} = [\eta_{i\text{-}1,2}, \boldsymbol{\epsilon}_{i\text{-}1,2}]^\top \in \mathbb{R}^4$ to represent the orientation of $\sum B_{i\text{-}1}$ relative to $\sum P_i$, and first establish the relationship between the angular displacements of $\sum B_i$ and $\sum B_{i\text{-}1}$ relative to $\sum P_i$ as follows. The Angle-axis representations of angular displacement of $\sum B_i$ and $\sum B_{i\text{-}1}$ relative to the $\sum P_i$, denoted as $\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}$ and $\Delta\theta_{i\text{-}1,2}\hat{\boldsymbol{e}}_{i\text{-}1,2}$, are presented in Fig. 1, where frame $\sum B_i$ and $\sum B_{i\text{-}1}$ rotates through the angle $\Delta\theta_{i,1}$ and $\Delta\theta_{i\text{-}1,2}$ about an axis defined by the unit vector $\hat{\boldsymbol{e}}_{i,1}$ and $\hat{\boldsymbol{e}}_{i\text{-}1,2}$ relative to frame $\sum P_i$, respectively. The following relationships exist between $\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}$ and $\Delta\theta_{i\text{-}1,2}\hat{\boldsymbol{e}}_{i\text{-}1,2}$ according to the relativity principle in motion and *rolling without slipping* condition between the two sphere bodies:

$$ \hat{\boldsymbol{e}}_{i\text{-}1,2} = -\hat{\boldsymbol{e}}_{i,1}, \tag{3} $$

and (refer to the Appendix A-A)

$$ l_i\dot{\theta}_{i\text{-}1,2} = l_{i\text{-}1}\dot{\theta}_{i,1}. \tag{4} $$

The relationships (3) and (4) result in

$$ \Delta\theta_{i\text{-}1,2}\hat{\boldsymbol{e}}_{i\text{-}1,2} = -\frac{l_{i\text{-}1}}{l_i}\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}. \tag{5} $$

Note that $\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}$ and $\Delta\theta_{i\text{-}1,2}\hat{\boldsymbol{e}}_{i\text{-}1,2}$ actually can be viewed as a geometry representation of $\Delta\boldsymbol{q}_{i,1}$ and $\Delta\boldsymbol{q}_{i\text{-}1,2}$ [8], where $\Delta\boldsymbol{q}_{i,1} = \boldsymbol{q}_{i,1}(t)\widetilde{\boldsymbol{q}}_{i,1}(t_0)$ is the angular displacement calculated with quaternions, similar for $\Delta\boldsymbol{q}_{i\text{-}1,2}$, and the symbol '$\widetilde{\square}$' represents the conjugate of a quaternion. Therefore, given initial state $\boldsymbol{q}_{i,1}(t_0)$ and $\boldsymbol{q}_{i,1}(t)$ integrated over Eq. (2), $\Delta\boldsymbol{q}_{i,1}$ can be calculated and converted to $\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}$. Then, $\Delta\theta_{i\text{-}1,2}\hat{\boldsymbol{e}}_{i\text{-}1,2}$ is calculated from $\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}$ using Eq. (5), and converted back to $\Delta\boldsymbol{q}_{i\text{-}1,2}$ and then calculates $\boldsymbol{q}_{i\text{-}1,2}(t)$ (the conversions between the Angle-axis representation and quaternion refer to the Appendix A-B).

So far, the orientations of the body $\sum B_i$ and $\sum B_{i\text{-}1}$ with respect to the tangent plane $\sum P_i$, represented by $\boldsymbol{q}_{i,1}$ and $\boldsymbol{q}_{i\text{-}1,2}$, have been obtained, and the orientation of $\sum B_i$ relative to $\sum B_{i\text{-}1}$, denoted as $\boldsymbol{q}_i$, can be calculated as

$$ \boldsymbol{q}_i = \widetilde{\boldsymbol{q}}_{i\text{-}1,2}\boldsymbol{q}_{i,1}. \tag{6} $$

Correspondingly, the rotation matrix of the *SRC* joint ${}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i}$ can be calculated from the unit quaternion $\boldsymbol{q}_i$ as shown in the Appendix A-C.

B. Position Vector

The position vector of the *SRC* joint is defined as the translations from the origin of body $\sum B_{i\text{-}1}$ to the origin of body $\sum B_i$ along the *SRC* joint axes.

As shown in Fig. 1, the position vector of the *SRC* joint can be expressed as

$$ {}^{B_{i\text{-}1}}_{P_i}\boldsymbol{r}_{B_i} = \begin{bmatrix} 0 \\ 0 \\ l_i + l_{i\text{-}1} \end{bmatrix}. \tag{7} $$

C. Free Modes

The free modes of the *SRC* joint define the directions in which the motion of $\sum B_i$ relative to $\sum B_{i\text{-}1}$ is allowed. They are represented by the $6 \times n_i$ matrix $\boldsymbol{\Phi}_i$ whose columns are the Plücker coordinates of the allowable motion [8], $n_i = 2$ is the number of the *SRC* joint DOFs. Defining the spatial velocity vector of $\sum B_i$, ${}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\nu}_{B_i} = \left[{}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i}^\top, {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i}^\top\right]^\top$, the matrix $\boldsymbol{\Phi}_i$ relates the spatial velocity vector ${}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\nu}_{B_i}$ to the *SRC* joint velocities ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}}$ and ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}}$ (note that the rotation about the vertical axis of the tangent plane P_i between the bodies is not allowed, i.e., ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iz}} = 0$):

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\nu}_{B_i} = \boldsymbol{\Phi}_i \begin{bmatrix} {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}} \\ {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}} \end{bmatrix}. \tag{8} $$

The angular velocities ${}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i}$ and ${}^{B_{i\text{-}1}}_{P_i}\boldsymbol{\omega}_{B_i}$ satisfy the relationship

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i} = {}^{B_{i\text{-}1}}\boldsymbol{R}_{P_i}\, {}^{B_{i\text{-}1}}_{P_i}\boldsymbol{\omega}_{B_i}. \tag{9} $$

Given the unit quaternion $\boldsymbol{q}_{i\text{-}1,2}$ in Section II-A, the rotation matrix ${}^{B_{i\text{-}1}}\boldsymbol{R}_{P_i}$ is obtained from $\boldsymbol{q}_{i\text{-}1,2}$:

$$ {}^{B_{i\text{-}1}}\boldsymbol{R}_{P_i} = \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}) & \boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2}) & \boldsymbol{f}_3(\boldsymbol{q}_{i\text{-}1,2}) \end{bmatrix}, \tag{10} $$

where the expressions of the functions $\boldsymbol{f}_i$ refer to the Appendix A-C.

Substituting Eq. (10) into Eq. (9) results in

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i} = \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}) & \boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2}) \end{bmatrix} \begin{bmatrix} {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}} \\ {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}} \end{bmatrix}. \tag{11} $$

Therefore, we obtain the angular free mode matrix $\boldsymbol{\Phi}_{i\boldsymbol{\omega}} = [\boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}),\ \boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2})]$ that describes the generated angular motion of $\sum B_i$ relative to $\sum B_{i\text{-}1}$ when a *SRC* joint velocity is given.

The linear velocity ${}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i}$ is calculated as (a more detailed derivation refers to the Appendix A-D)

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i} = (l_i + l_{i\text{-}1})\boldsymbol{f}_3^{\times}(\boldsymbol{q}_{i\text{-}1,2})\left({}^{P_i}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_{i\text{-}1}}\right), \tag{12} $$

where $\boldsymbol{f}_3^{\times}(\boldsymbol{q}_{i\text{-}1,2})$ is the skew-symmetric matrix of $\boldsymbol{f}_3(\boldsymbol{q}_{i\text{-}1,2})$.

Similar to the *SRC* joint velocity relationship in Eq. (1), one has

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i} = -\frac{l_i + l_{i\text{-}1}}{l_{i\text{-}1}}\,{}^{P_i}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_{i\text{-}1}}, \tag{13} $$

which makes Eq. (12) be

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i} = -l_{i\text{-}1}\boldsymbol{f}_3^{\times}(\boldsymbol{q}_{i\text{-}1,2})\,{}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i}, \tag{14} $$

and using Eq. (11), Eq. (14) turns into

$$ \begin{aligned} &{}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i} = \\ &-l_{i\text{-}1}\left[\boldsymbol{f}_3^{\times}\boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}), \boldsymbol{f}_3^{\times}\boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2})\right] \begin{bmatrix} {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}} \\ {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}} \end{bmatrix}. \end{aligned} \tag{15} $$

Therefore, the linear free mode matrix $\boldsymbol{\Phi}_{i\boldsymbol{v}} = [-l_{i\text{-}1}\boldsymbol{f}_3^{\times}\boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}),\ -l_{i\text{-}1}\boldsymbol{f}_3^{\times}\boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2})]$ is obtained, which describes the generated linear motion of $\sum B_i$ relative to $\sum B_{i\text{-}1}$ when a *SRC* joint velocity is given.

Combining Eqs. (11) and (15), the free modes matrix $\boldsymbol{\Phi}_i$ of the *SRC* joint in Eq. (8) can be expressed as

$$ \boldsymbol{\Phi}_i = \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}) & \boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2}) \\ -l_{i\text{-}1}\boldsymbol{f}_3^{\times}\boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}) & -l_{i\text{-}1}\boldsymbol{f}_3^{\times}\boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2}) \end{bmatrix}, \tag{16} $$

which reflects the spatial velocity of the body $\sum B_i$ relative to the body $\sum B_{i\text{-}1}$ due to the *SRC* joint velocity.

As a summary, Section II establishes the *SRC* joint model, which describes the motion of frame $\sum B_i$ fixed in one body of the *SRC* joint relative to frame $\sum B_{i\text{-}1}$ fixed in the other body. The motion is expressed as a function of the *SRC* joint motion variables $\boldsymbol{q}_i$ and ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}}$, ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}}$, and other elements of the *SRC* joint. As with other classic robot joints (Table 2.5, 2.6 in [8]), the *SRC* joint's elements include the rotation matrix ${}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i}$, position vector ${}^{B_{i\text{-}1}}_{P_i}\boldsymbol{r}_{B_i}$, and free modes $\boldsymbol{\Phi}_i$, which have been summarized and listed in Table I. The elements are also important for deriving the kinematics model of the robot manipulator composed of the *SRC* joints in Section III.

TABLE I: Joint model formulas for 2-DOF *SRC* joint

Joint type	2-DOF *SRC*
Rotation matrix, ${}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i}$	(see Eq.(6))
Position vector, ${}^{B_{i\text{-}1}}_{P_i}\boldsymbol{r}_{B_i}$	$\begin{bmatrix} 0 \\ 0 \\ l_i + l_{i\text{-}1} \end{bmatrix}$
Free modes, $\boldsymbol{\Phi}_i$	$\begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{q}_{i-1,2}) & \boldsymbol{f}_2(\boldsymbol{q}_{i-1,2}) \\ l_{i\text{-}1}\boldsymbol{f}_3^{\times}\boldsymbol{f}_1(\boldsymbol{q}_{i-1,2}) & l_{i\text{-}1}\boldsymbol{f}_3^{\times}\boldsymbol{f}_2(\boldsymbol{q}_{i-1,2}) \end{bmatrix}$
Pose variables	$\boldsymbol{q}_i$
Velocity variables	$\begin{bmatrix} {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}} \\ {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}} \end{bmatrix}$

III. Kinematics Modeling of *SRC* Joint Manipulator

In this section, a serial robot manipulator composed of *SRC* joints is presented as shown in Fig. 2, with its forward and inverse kinematics being thoroughly modeled.

Three kinds of coordinate frame are defined to describe relative motions among the manipulator bodies: inertial frame $\sum I$, link body-fixed frame $\sum B_i$, and virtual tangent plane body-fixed frame $\sum P_i$ (also as the joint frame), where $i = 1, \cdots, n$, n is the number of the joints/links. B_0 represents the base of the manipulator, the last link B_n is treated as the end-effector. Define the position of the geometry center of the spherical link B_i in the inertial frame $\sum I$ as ${}^I_I\boldsymbol{r}_{B_i}$, the orientation of link B_i relative to $\sum I$ as the unit quaternion $\boldsymbol{q}_{i,I}$, and the radius of the link B_i as l_i.

A. Forward kinematics

The *forward kinematics* of the serial *SRC* joint manipulator determines the pose (i.e., position and orientation) of the end-effector, ${}^I_I\boldsymbol{r}_{B_n}$ and $\boldsymbol{q}_{n,I}$, (or its linear and angular velocities, ${}^I_I\boldsymbol{v}_{B_n}$ and ${}^I_I\boldsymbol{\omega}_{B_n}$, in the *forward instantaneous kinematics* problem), given the positions $\boldsymbol{q}_i$ (or velocities ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}}$, ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}}$) of all of the *SRC* joints and the values of all of the geometric link parameters l_i.

For the *SRC* joint manipulator, it exists that

$$ {}^I\boldsymbol{R}_{B_n} = {}^I\boldsymbol{R}_{B_0}\prod_{i=1}^{n}\left({}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i}\right). \tag{17} $$

Given the *SRC* joints positions $\boldsymbol{q}_i$, the rotation matrices of the *SRC* joints ${}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i}$ are calculated as shown in Section II-A. Without loss of generality, ${}^I\boldsymbol{R}_{B_0}$ is the identity matrix when the base B_0 is fixed. Therefore, the rotation matrix of the end-effector with respect to the inertial frame, ${}^I\boldsymbol{R}_{B_n}$, can be calculated from $\boldsymbol{q}_i$. Then, the orientation of the end-effector relative to the inertial frame, $\boldsymbol{q}_{n,I}$, can be extracted from ${}^I\boldsymbol{R}_{B_n}$ [8].

In order to calculate the angular velocity of the end-effector in the inertial frame ${}^I_I\boldsymbol{\omega}_{B_n}$ when given the velocities ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}}$, ${}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}}$ of all of the *SRC* joints, according to the free modes of the *SRC* joint $\boldsymbol{\Phi}_i$ in Table I, one first has

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i} = \boldsymbol{\Phi}_{i\boldsymbol{\omega}}\begin{bmatrix} {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}} \\ {}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}} \end{bmatrix}. \tag{18} $$

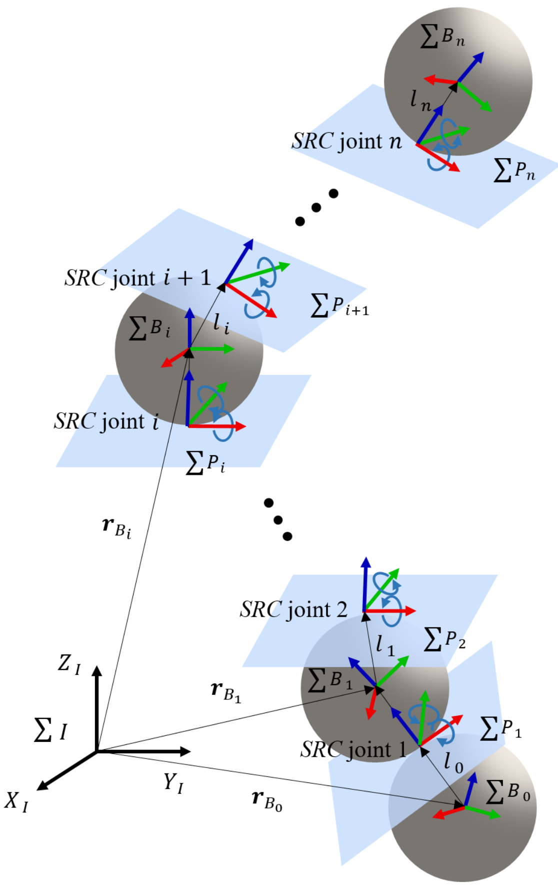


Fig. 2: Diagram of the serial *SRC* joint manipulator

Then, the angular velocity ${}^{I}_{I}\boldsymbol{\omega}_{B_i}, i = 1, \cdots, n$ can be iteratively calculated as (iterative process refers to the Appendix B-A)

$$ {}^{I}_{I}\boldsymbol{\omega}_{B_i} = \sum_{j=1}^{i}\left({}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\boldsymbol{\Phi}_{j\boldsymbol{\omega}}\begin{bmatrix}{}^{B_{j\text{-}1}}_{\ \ P_j}\omega_{B_{jx}}\\ {}^{B_{j\text{-}1}}_{\ \ P_j}\omega_{B_{jy}}\end{bmatrix}\right) + {}^{I}_{I}\boldsymbol{\omega}_{B_0}. \quad (19) $$

Specially, one has

$$ {}^{I}_{I}\boldsymbol{\omega}_{B_n} = \sum_{i=1}^{n}\left({}^{I}\boldsymbol{R}_{B_{i\text{-}1}}\boldsymbol{\Phi}_{i\boldsymbol{\omega}}\begin{bmatrix}{}^{B_{i\text{-}1}}_{\ \ P_i}\omega_{B_{ix}}\\ {}^{B_{i\text{-}1}}_{\ \ P_i}\omega_{B_{iy}}\end{bmatrix}\right) + {}^{I}_{I}\boldsymbol{\omega}_{B_0}. \quad (20) $$

By defining the involved *SRC* joint velocity set

$$ \boldsymbol{\omega} = \begin{bmatrix}{}^{B_0}_{P_1}\omega_{B_{1x}}\\ {}^{B_0}_{P_1}\omega_{B_{1y}}\\ \vdots\\ {}^{B_{n\text{-}1}}_{\ \ P_n}\omega_{B_{nx}}\\ {}^{B_{n\text{-}1}}_{\ \ P_n}\omega_{B_{ny}}\end{bmatrix} \in \mathbb{R}^{2n}, \quad (21) $$

as in Eqs. (19) and (20), the *forward angular velocity equations* of the *SRC* joint manipulator can be obtained as follows

$$ {}^{I}_{I}\boldsymbol{\omega}_{B_i} = {}^{I}_{I}\boldsymbol{\omega}_{B_0} + \boldsymbol{J}_{B_i\boldsymbol{\omega}}\,\boldsymbol{\omega}, \quad (22) $$
$$ {}^{I}_{I}\boldsymbol{\omega}_{B_n} = {}^{I}_{I}\boldsymbol{\omega}_{B_0} + \boldsymbol{J}_{B_n\boldsymbol{\omega}}\,\boldsymbol{\omega}, \quad (23) $$

where $\boldsymbol{J}_{B_i\boldsymbol{\omega}}, \boldsymbol{J}_{B_n\boldsymbol{\omega}}$ are termed as the angular Jacobian matrices of the *SRC* joint manipulator that map the joint velocities into the angular velocities of the link i and end-effector, and

$$ \boldsymbol{J}_{B_i\boldsymbol{\omega}} = \left[{}^{I}\boldsymbol{R}_{B_0}\boldsymbol{\Phi}_{1\boldsymbol{\omega}} \ \cdots \ {}^{I}\boldsymbol{R}_{B_{i\text{-}1}}\boldsymbol{\Phi}_{i\boldsymbol{\omega}} \ \ \boldsymbol{0} \ \cdots \ \boldsymbol{0}\right] \in \mathbb{R}^{3\times 2n}, \quad (24) $$

$$ \boldsymbol{J}_{B_n\boldsymbol{\omega}} = \left[{}^{I}\boldsymbol{R}_{B_0}\boldsymbol{\Phi}_{1\boldsymbol{\omega}} \ \cdots \ {}^{I}\boldsymbol{R}_{B_{n\text{-}1}}\boldsymbol{\Phi}_{n\boldsymbol{\omega}}\right] \in \mathbb{R}^{3\times 2n}. \quad (25) $$

The positions ${}^{I}_{I}\boldsymbol{r}_{B_i}$ and ${}^{I}_{I}\boldsymbol{r}_{B_n}$ are calculated as follows

$$ {}^{I}_{I}\boldsymbol{r}_{B_i} = {}^{I}_{I}\boldsymbol{r}_{B_0} + \sum_{j=1}^{i}\left({}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\,{}^{B_{j\text{-}1}}_{B_{j\text{-}1}}\boldsymbol{r}_{B_j}\right), \quad (26) $$

$$ {}^{I}_{I}\boldsymbol{r}_{B_n} = {}^{I}_{I}\boldsymbol{r}_{B_0} + \sum_{i=1}^{n}\left({}^{I}\boldsymbol{R}_{B_{i\text{-}1}}\,{}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{r}_{B_i}\right). \quad (27) $$

According to the free modes of the *SRC* joint $\boldsymbol{\Phi}_i$ in Table I, one has

$$ {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i} = \boldsymbol{\Phi}_{i\boldsymbol{v}}\begin{bmatrix}{}^{B_{i\text{-}1}}_{\ \ P_i}\omega_{B_{ix}}\\ {}^{B_{i\text{-}1}}_{\ \ P_i}\omega_{B_{iy}}\end{bmatrix}. \quad (28) $$

Differentiating Eq. (26) and using the relations (22) and (28) results in (a more detailed derivation refers to the Appendix B-B)

$$ {}^{I}_{I}\boldsymbol{v}_{B_i} = {}^{I}_{I}\boldsymbol{v}_{B_0} + {}^{I}_{I}\boldsymbol{r}^{\times}_{0i}\,{}^{I}_{I}\boldsymbol{\omega}_{B_0} + \boldsymbol{J}_{B_i\boldsymbol{v}}\boldsymbol{\omega}, \quad (29) $$

where $\boldsymbol{J}_{B_i\boldsymbol{v}}$ is termed as the linear Jacobian matrix of the *SRC* joint manipulator that maps the joint velocities into the linear velocity of the geometry center of the link i, and

$$ {}^{I}_{I}\boldsymbol{r}_{0i} = {}^{I}_{I}\boldsymbol{r}_{B_0} - {}^{I}_{I}\boldsymbol{r}_{B_i} = -\sum_{j=1}^{i}{}^{B_{j\text{-}1}}_{\ \ I}\boldsymbol{r}_{B_j} \in \mathbb{R}^{3}, \quad (30) $$

$$ \boldsymbol{J}'_{B_i\boldsymbol{v}} = \left[{}^{I}\boldsymbol{R}_{B_0}\boldsymbol{\Phi}_{1\boldsymbol{v}} \ \cdots \ {}^{I}\boldsymbol{R}_{B_{i\text{-}1}}\boldsymbol{\Phi}_{i\boldsymbol{v}} \ \ \boldsymbol{0} \ \cdots \ \boldsymbol{0}\right] \in \mathbb{R}^{3\times 2n}, \quad (31) $$

$$ \boldsymbol{J}_{B_i\boldsymbol{v}} = \boldsymbol{J}'_{B_i\boldsymbol{v}} - \sum_{j=1}^{i}\left({}^{B_{j\text{-}1}}_{\ \ I}\boldsymbol{r}^{\times}_{B_j}\boldsymbol{J}_{B_{j\text{-}1}\boldsymbol{\omega}}\right) \in \mathbb{R}^{3\times 2n}, \quad (32) $$

Similarly, Eq. (27) can be derived and results in

$$ {}^{I}_{I}\boldsymbol{v}_{B_n} = {}^{I}_{I}\boldsymbol{v}_{B_0} + {}^{I}_{I}\boldsymbol{r}^{\times}_{0n}\,{}^{I}_{I}\boldsymbol{\omega}_{B_0} + \boldsymbol{J}_{B_n\boldsymbol{v}}\boldsymbol{\omega}, \quad (33) $$

where $\boldsymbol{J}_{B_n\boldsymbol{v}}$ is the linear Jacobian matrix that maps the joint velocities into the linear velocity of the end-effector.

$$ {}^{I}_{I}\boldsymbol{r}_{0n} = {}^{I}_{I}\boldsymbol{r}_{B_0} - {}^{I}_{I}\boldsymbol{r}_{B_n} = -\sum_{i=1}^{n}{}^{B_{i\text{-}1}}_{\ \ I}\boldsymbol{r}_{B_i} \in \mathbb{R}^{3}, \quad (34) $$

$$ \boldsymbol{J}'_{B_n\boldsymbol{v}} = \left[{}^{I}\boldsymbol{R}_{B_0}\boldsymbol{\Phi}_{1\boldsymbol{v}} \ \cdots \ {}^{I}\boldsymbol{R}_{B_{n\text{-}1}}\boldsymbol{\Phi}_{n\boldsymbol{v}}\right] \in \mathbb{R}^{3\times 2n}, \quad (35) $$

$$ \boldsymbol{J}_{B_n\boldsymbol{v}} = \boldsymbol{J}'_{B_n\boldsymbol{v}} - \sum_{i=1}^{n}\left({}^{B_{i\text{-}1}}_{\ \ I}\boldsymbol{r}^{\times}_{B_i}\boldsymbol{J}_{B_{i\text{-}1}\boldsymbol{\omega}}\right) \in \mathbb{R}^{3\times 2n}, \quad (36) $$

Eqs. (29) and (33) present the *forward linear velocity equations* of the *SRC* joint manipulator.

B. Inverse kinematics

The *inverse kinematics* (IK) problem for the *SRC* joint manipulator is to find the values of the *SRC* joint positions $\boldsymbol{q}_i$ (or velocities ${}^{B_{i\text{-}1}}_{\ \ P_i}\omega_{B_{ix}}$, ${}^{B_{i\text{-}1}}_{\ \ P_i}\omega_{B_{iy}}$ in the *inverse instantaneous kinematics* problem) given the pose, ${}^{I}_{I}\boldsymbol{r}_{B_n}$ and $\boldsymbol{q}_{n,I}$, (or linear and angular velocities, ${}^{I}_{I}\boldsymbol{v}_{B_n}$ and ${}^{I}_{I}\boldsymbol{\omega}_{B_n}$) of the end-effector relative to the inertial frame and the values of all of the geometric link parameters l_i.

Given the forward kinematics equations of the *SRC* joint manipulator, Eqs. (17) and (27), it is clear that the IK problem for the *SRC* joint manipulator requires the solution of sets of nonlinear equations, and is difficult to be presented in closed form. This section presents the damped least squares (DLS) based IK method for the *SRC* joint manipulator, which first obtains the smooth *SRC* joint velocities that correspond to a desired end-effector velocity, i.e., the inverse instantaneous kinematics problem of the *SRC* joint manipulator is solved. Then, the *SRC* joint positions, which correspond to the specific pose of the end-effector, can be obtained by performing the joint velocities integration resulted from the velocities of the end-effector that lead to its desired pose and orientation.

In order to obtain the required *SRC* joint velocities $\boldsymbol{\omega}$ that guarantee the desired linear and angular velocities of the end-effecotr simultaneously, the *forward linear and angular velocity equations* (23) and (33) are combined together:

$$\begin{bmatrix} {}_I^I\boldsymbol{v}_{B_n} \\ {}_I^I\boldsymbol{\omega}_{B_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_3 & {}_I^I\boldsymbol{r}_{0n}^{\times} \\ \boldsymbol{0} & \boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} {}_I^I\boldsymbol{v}_{B_0} \\ {}_I^I\boldsymbol{\omega}_{B_0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{J}_{B_n\boldsymbol{v}} \\ \boldsymbol{J}_{B_n\boldsymbol{\omega}} \end{bmatrix} \boldsymbol{\omega}, \tag{37}$$

where $\boldsymbol{I}_3$ is the identity matrix.

Without loss of generality, this paper supposes that the *SRC* joint manipulator has a fixed base, Eq. (37) can be simplified as

$$\begin{bmatrix} {}_I^I\boldsymbol{v}_{B_n} \\ {}_I^I\boldsymbol{\omega}_{B_n} \end{bmatrix} = \boldsymbol{J}_{B_n}\boldsymbol{\omega}, \tag{38}$$

where $\boldsymbol{J}_{B_n} = [\boldsymbol{J}_{B_n\boldsymbol{v}}^\top, \boldsymbol{J}_{B_n\boldsymbol{\omega}}^\top]^\top \in \mathbb{R}^{6\times 2n}$ is termed as the Jacobian matrix of the *SRC* joint manipulator.

Using the adaptive DLS method to solve the kinematics equations (38), one performs singular-value-decomposition on the Jacobian matrix $\boldsymbol{J}_{B_n}$ [28]

$$\boldsymbol{J}_{B_n} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top, \tag{39}$$

where $\boldsymbol{U} \in \mathbb{R}^{6\times 6}$ and $\boldsymbol{V} \in \mathbb{R}^{2n\times 2n}$ are unitary matrices composed of the left and right-singular vectors of $\boldsymbol{J}_{B_n}$. $\boldsymbol{\Sigma} \in \mathbb{R}^{6\times 2n}$ is a rectangular diagonal matrix with singular values of $\boldsymbol{J}_{B_n}$ on the diagonal, which is calculated as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \sigma_6 & 0 & \cdots & 0 \end{bmatrix}, \text{ for } 6 \leq 2n \tag{40}$$

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{2n} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \text{ for } 6 > 2n \tag{41}$$

with σ_i being the singular values ordered so that $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_{\min(6,2n)} = 0$, r is the rank of $\boldsymbol{J}_{B_n}$.

Using Eq. (39), the adaptive DLS based IK solution of Eq. (38) is [29, 30]

$$\boldsymbol{\omega} = \boldsymbol{V}\boldsymbol{\Sigma}^*\boldsymbol{U}^\top \begin{bmatrix} {}_I^I\boldsymbol{v}_{B_n} \\ {}_I^I\boldsymbol{\omega}_{B_n} \end{bmatrix}, \tag{42}$$

where $\boldsymbol{\Sigma}^* \in \mathbb{R}^{2n\times 6}$ is the pseudoinverse of $\boldsymbol{\Sigma}$ and formed by replacing every none-zero diagonal entry σ_i of $\boldsymbol{\Sigma}$ with its damped reciprocal

$$\sigma_i^* = \frac{\sigma_i}{\sigma_i^2 + \lambda^2}, \; i = 1, \cdots, r, \tag{43}$$

and transposing the resulting matrix. The damping factor λ^2 is adaptively adopted according to the following law:

$$\lambda^2 = \begin{cases} 0, & \text{if } \sigma_{\min(6,2n)} \geq \xi \\ \left(1 - \left(\dfrac{\sigma_r}{\xi}\right)^2\right)\lambda_{\max}^2, & \text{otherwise} \end{cases} \tag{44}$$

where σ_r is the smallest positive singular value of $\boldsymbol{J}_{B_n}$, and ξ defines the size of the singular region; the value of $\lambda_{\max}$ is at user's disposal to suitably shape the solution in the neighborhood of a singularity.

The smooth *SRC* joint velocity $\boldsymbol{\omega}$ that corresponds to the specific end-effector velocities ${}_I^I\boldsymbol{v}_{B_n}, {}_I^I\boldsymbol{\omega}_{B_n}$ can be obtained using Eq. (42). In order to solve the joint positions $\boldsymbol{q}_i$ that corresponds to the desired pose of the end-effector, ${}_I^I\boldsymbol{r}_{B_n}$ and $\boldsymbol{q}_{n,I}$, a closed-loop reference velocity profile ${}_I^I\boldsymbol{v}_{B_n}^r$ and ${}_I^I\boldsymbol{\omega}_{B_n}^r$ for the end-effector reaching the desired pose is designed as follows

$${}_I^I\boldsymbol{v}_{B_n}^r = {}_I^I\dot{\boldsymbol{r}}_{B_n}^d + \boldsymbol{K}_p({}_I^I\boldsymbol{r}_{B_n}^d - {}_I^I\boldsymbol{r}_{B_n}), \tag{45}$$

$${}_I^I\boldsymbol{\omega}_{B_n}^r = {}_I^I\boldsymbol{\omega}_{B_n}^d + \boldsymbol{K}_a\boldsymbol{\delta}_{B_n,\text{Quat}}, \tag{46}$$

$$\boldsymbol{\delta}_{B_n,\text{Quat}} = \eta_{n,I}\boldsymbol{\epsilon}_{n,I}^d - \eta_{n,I}^d\boldsymbol{\epsilon}_{n,I} - \boldsymbol{\epsilon}_{n,I}^d \times \boldsymbol{\epsilon}_{n,I}, \tag{47}$$

where the right-superscript 'd' denotes the desired values, $\eta_{n,I}$ and $\boldsymbol{\epsilon}_{n,I}$ are the scalar and vector parts of $\boldsymbol{q}_{n,I}$ respectively, and the resulting end-effector orientation errors can be calculated using Eq. (47). $\boldsymbol{K}_p, \boldsymbol{K}_a \in \mathbb{R}^{3\times 3}$ are suitable positive definite matrix gains. It is readily to prove that the end-effector pose errors globally asymptotically converge to zeros by following the velocity profile ${}_I^I\boldsymbol{v}_{B_n}^r$ and ${}_I^I\boldsymbol{\omega}_{B_n}^r$ [31]. Given the end-effector reference velocities ${}_I^I\boldsymbol{v}_{B_n}^r$ and ${}_I^I\boldsymbol{\omega}_{B_n}^r$, the corresponding *SRC* joint reference velocities $\boldsymbol{\omega}^r$ are solved using Eq. (42). Then, the required joint positions $\boldsymbol{q}_i^r$ for reaching the desired end-effector pose are obtained by performing the joint reference velocity integration using the *SRC* joint model in Section II.

C. Discussion

The *SRC* joint manipulator has the advantages of the enlargement of motion range, the absence of abrasion wear, provision of a transmission ratio directly at the joint, etc. It is worth noting that due to the geometric symmetry of the spherical units, the *SRC* joints can roll forward flexibly in all directions, such that the manipulator can easily perform tasks in environments with static and dynamic obstacles, as well as in narrow spaces. The drive of the *SRC* joint can also be novelly physically implemented. Unlike conventional

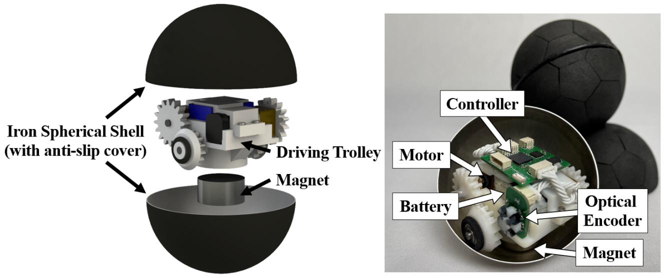


Fig. 3: The components of a FreeBOT module

robotic manipulator that use combinations of revolute and prismatic joints implemented by DC motors, the realization of the *SRC* joint could be using the FreeBOT for example [27], as shown in detail in Section IV, where a driving trolley moves inside the spherical iron body and uses a magnet at trolley's bottom to attract the other body not separated. In turn, performing chain-type motion is common for a MSRR robot (like FreeBOT) in self-reconfiguration tasks [32], the established *SRC* joint model and manipulator kinematics are also useful for describing the reconfiguration motions of the FreeBOT.

IV. Control of *SRC* joint manipulator

A MSRR robot whose module can be freely connected, termed as FreeBOT, has been proposed by Liang et al. [27] This section uses FreeBOTs to realize the kinematic motion of the 2-DOF *SRC* joint and manipulator composed of *SRC* joints. The desired *SRC* joint motions are translated into those of the driving trolley inside the FreeBOT through path planning, then control method is proposed for the trolley, such that the trolley drives the *SRC* joint and end-effector of the manipulator to move as expected.

A. FreeBOT components

As shown in Fig. 3, a FreeBOT is mainly composed of an iron sphere shell, a differential-wheel driven trolley inside the shell, a permanent magnet at the trolley bottom, and other measurement and control units. When a FreeBOT moves over another one, the magnetic force between the permanent magnet of one FreeBOT and the iron shell of the other FreeBOT guarantees the two FreeBOTs not separated. A layer of EVA foam is applied to the outer surface of the iron sphere shell to increase friction forces and torques, such that the two FreeBOTs roll relatively without slipping at the contact point, and the rotation about the vertical axis of the tangent plane at the contact point is avoided.

Obviously, the 2-DOF *SRC* joint is singularity-free when the driving trolley is holonomic, for instance, using omni-directional wheels. Differential-wheel driven trolley, on the other hand, are widely used in robotics owing to its simplicity. In a FreeBOT, the trolley *steers* freely inside the shell about the vertical axis of the tangent plane, and *moves forward* along arbitrary direction inside the tangent plane, thus the 2-DOF motions of the *SRC* joint are realized. The relationships

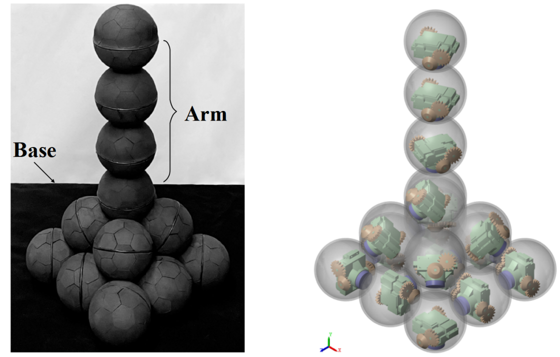


Fig. 4: FreeBOT realization of the *SRC* joint manipulator

between the trolley steering and forward velocities and the *SRC* joint velocities ${}^{B_{i-1}}_{P_i}\boldsymbol{\omega}_{B_i}$ are discussed in Section IV-B.

In this paper, a Lyapunov-based control method is proposed for the non-holonomic differential-wheel driven trolley to realize the required motions of the *SRC* joint and manipulator. By taking the *first steering* to the desired angle then *moving forward* strategy can also simulate the control effect of a holonomic trolley, i.e., simultaneously generating accurate 2-DOF angular velocity of the *SRC* joint, which will be explained in detail in Section IV-C.

B. Trolley path planning

A serial-chain *SRC* joint manipulator formed by FreeBOTs is presented in Fig. 4. For the end-effector (assumed as the shell of the end FreeBOT) moving with specific velocities or reaching a desired pose, the corresponding *SRC* joint reference velocities $\boldsymbol{\omega}^r$ or final joint positions $\boldsymbol{q}_i^r$ have been obtained through solving the IK problem in Section III-B. This section determines the desired motions of the driving trolleys, such that the corresponding *SRC* joint can have the required velocity ${}^{B_{i-1}}_{P_i}\omega^r_{B_{ix}}$, ${}^{B_{i-1}}_{P_i}\omega^r_{B_{iy}}$ or reach the required joint position $\boldsymbol{q}_i^r$.

1) velocity planning: For the *SRC* joint generating the required angular velocity ${}^{B_{i-1}}_{P_i}\omega^r_{B_{ix}}$, ${}^{B_{i-1}}_{P_i}\omega^r_{B_{iy}}$, we define the *forward* and *steering* speed of the driving trolley i as u_{if} and u_{is}, respectively, denote the angular displacement of the trolley i caused by the steering motion as φ_i, the linear velocity of the trolley i in the tangent plane $\sum P_i$, ${}^{P_i}_{P_i}\boldsymbol{v}_{T_i}$, can be calculated as follows

$$ {}^{P_i}_{P_i}\boldsymbol{v}_{T_i} = \begin{bmatrix} u_{if}\cos\varphi_i \\ u_{if}\sin\varphi_i \\ 0 \end{bmatrix}, \tag{48}$$

with the steering dynamics

$$\dot{\varphi}_i = u_{is}, \tag{49}$$

where $\varphi_i = 0$ corresponds to that the forward direction of the trolley i points towards the $\hat{x}$ axis of $\sum P_i$.

According to the non-holonomic constraint that *rolling without slipping* occurs at the contact point between the two FreeBOTs, the following relationship exists

$$\begin{cases} {}^{P_i}_{P_i}v_{T_{ix}} - l_i\, {}^{P_i}_{P_i}\omega_{B_{iy}} = 0, \\ {}^{P_i}_{P_i}v_{T_{iy}} + l_i\, {}^{P_i}_{P_i}\omega_{B_{ix}} = 0. \end{cases} \tag{50}$$

Substituting Eq. (48) and the *SRC* joint velocity relationship (1) into Eq. (50) results in

$$\begin{cases} u_{if}\cos\varphi_i - l_i'\,{}^{B_{i\text{-}1}}_{P_i}\omega_{B_{iy}} = 0, \\ u_{if}\sin\varphi_i + l_i'\,{}^{B_{i\text{-}1}}_{P_i}\omega_{B_{ix}} = 0, \end{cases} \tag{51}$$

where $l_i' = l_i\, l_{i\text{-}1}/(l_i + l_{i\text{-}1})$.

Using Eq. (51), in order to generate the required *SRC* joint velocity ${}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{ix}}, {}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{iy}}$, the driving trolley i should move with the forward speed u^r_{if} along the specific direction φ^r_i, and u^r_{if}, φ^r_i are calculated as

$$\begin{cases} u^r_{if} = l_i'\sqrt{\left({}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{ix}}\right)^2 + \left({}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{iy}}\right)^2}, \\ \varphi^r_i = \text{atan2}\left(-\,{}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{ix}} / \,{}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{iy}}\right). \end{cases} \tag{52}$$

2) position planning: For the *SRC* joint reaching the required joint position $\boldsymbol{q}^r_i$, the corresponding orientation of $\sum P_i$ relative to $\sum B_{i\text{-}1}$, represented by the unit quaternion $\widetilde{\boldsymbol{q}}^r_{i\text{-}1,2}$, is determined from $\boldsymbol{q}^r_i$ using the *SRC* joint model in Section II. The rotation matrix ${}^{B_{i\text{-}1}}\boldsymbol{R}^r_{P_i}$ is calculated from $\widetilde{\boldsymbol{q}}^r_{i\text{-}1,2}$. Therefore, the position of the contact point between the two FreeBOTs in the $\sum B_{i\text{-}1}$, which corresponds to the required *SRC* joint position $\boldsymbol{q}^r_i$, is determined:

$$\begin{aligned} {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{r}^r_{P_i} &= {}^{B_{i\text{-}1}}\boldsymbol{R}^r_{P_i}\,{}^{B_{i\text{-}1}}_{P_i}\boldsymbol{r}_{P_i} \\ &= {}^{B_{i\text{-}1}}\boldsymbol{R}^r_{P_i}\,[0, 0, l_{i\text{-}1}]^\top, \end{aligned} \tag{53}$$

which is constant, and can be translated into the moving frame $\sum P_i$ as

$${}^{P_i}_{P_i}\boldsymbol{r}^r_{P_i} = {}^{P_i}\boldsymbol{R}_{B_{i\text{-}1}}({}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{r}^r_{P_i} - {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{r}_{P_i}). \tag{54}$$

It is intuitive that the target position of the contact point, ${}^{P_i}_{P_i}\boldsymbol{r}^r_{P_i}$, can be expressed as going through a circular arc from its current position ${}^{P_i}_{P_i}\boldsymbol{r}_{P_i}$ in $\sum P_i$, given that the contact point always moves on the sphere surface of the FreeBOT $B_{i\text{-}1}$. There are two parameters to describe the reference circular arc, the arc distance s^r_i and the arc azimuth φ^r_i, which are calculated as

$$\begin{cases} s^r_i = l_{i\text{-}1}\,\text{acos}\left(\left(l_{i\text{-}1} - \text{abs}\left({}^{P_i}_{P_i}\boldsymbol{r}^r_{P_iz}\right)\right)/l_{i\text{-}1}\right), \\ \varphi^r_i = \text{atan2}\left({}^{P_i}_{P_i}\boldsymbol{r}^r_{P_ix} / \,{}^{P_i}_{P_i}\boldsymbol{r}^r_{P_iy}\right). \end{cases} \tag{55}$$

Therefore, in order to reach the required *SRC* joint position $\boldsymbol{q}^r_i$, the driving trolley i should go through the arc distance s^r_i along the specific direction φ^r_i in the tangent plane.

C. Trolley control method

Herein, the forward and steering speeds of the driving trolley u_{if} and u_{is}, as well as the rotation speeds of the two differential wheels w_{ir} and w_{il}, are sought, such that the driving trolley can move forward with the reference speed u^r_{if} or move through the reference arc distance s^r_i along the reference direction φ^r_i. As a result, the *SRC* joint (and correspondingly the end-effector) can have the reference velocities or reach the desired poses, respectively.

1) forward and steering dependent: Following the steering dynamics of the trolley i in Eq. (49), the following trolley steering speed guarantees that the steering angle converges to the reference direction φ^r_i:

$$u^r_{is} = \dot{\varphi}^r_i + k_{\varphi_{i,1}}(\varphi^r_i - \varphi_i). \tag{56}$$

In order to make the trolley i moves forward with the speed u^r_{if} along the reference direction φ^r_i, which generates the reference *SRC* joint velocity ${}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{ix}}$ and ${}^{B_{i\text{-}1}}_{P_i}\omega^r_{B_{iy}}$ (as shown in Eq. (52)), it is designed that the trolley moves forward and steers simultaneously along the velocity

$$\begin{cases} u_{if} = u^r_{if}\cos\varphi_{ie}, \\ u_{is} = u^r_{is} + k_{\varphi_{i,2}}u^r_{if}\sin\varphi_{ie}, \end{cases} \tag{57}$$

where φ_{ie} is the steering angle error, $k_{\varphi_{i,1}}, k_{\varphi_{i,2}}$ are positive constant gains.

Using Eq. (57) as the control rule of the trolley i, $\varphi_{ie} = 0$ is a uniformly asymptotically stable point. In this case, from Eq. (57), it is guaranteed that $u_{if} = u^r_{if}$, which means the trolley i can move forward with the speed u^r_{if} along the reference direction φ^r_i. As a result, the *SRC* joints (and correspondingly the end-effector) can have the reference velocities.

Proof. A scalar function V as a Lyapunov function candidate is proposed:

$$V = 1 - \cos\varphi_{ie}. \tag{58}$$

Clearly, $V \geq 0$. If $\varphi_{ie} = 0$, $V = 0$. If $\varphi_{ie} \neq 0$, $V > 0$.

Substituting the control rule Eq. (57) into the steering dynamics Eq. (49), one obtains the closed-loop steering dynamics

$$\dot{\varphi}_{ie} = -k_{\varphi_{i,2}}u^r_{if}\sin\varphi_{ie}. \tag{59}$$

Therefore, one has

$$\begin{aligned} \dot{V} &= \dot{\varphi}_{ie}\sin\varphi_{ie} \\ &= -k_{\varphi_{i,2}}u^r_{if}\sin^2\varphi_{ie} \leq 0. \end{aligned} \tag{60}$$

Then, V becomes a Lyapunov function.

By linearizing the differential Eq. (59) around $\varphi_{ie} = 0$,

$$\dot{\varphi}_{ie} + k_{\varphi_{i,2}}u^r_{if}\varphi_{ie} = 0. \tag{61}$$

It is readily to see that $k_{\varphi_{i,2}}u^r_{if}$ is continuously differentiable and is bounded. The characteristic equation for Eq. (61) is

$$s + k_{\varphi_{i,2}}u^r_{if} = 0. \tag{62}$$

From Eq. (52), there exits a positive constant δ such that $u^r_{if} \geq \delta$ for all $t \geq 0$. Therefore, the root of Eq. (62) have real parts less than or equal to $-k_{\varphi_{i,2}}\delta$, which implies that $\varphi_{ie} = 0$ is a uniformly asymptotically stable point of Eq. (61). □

2) forward and steering independent: In order to simulate the control effect of a holonomic driving trolley, i.e., simultaneously generating the accurate 2-DOF *SRC* joint velocity, we adopt the *first steering* to the desired angle then *moving forward* strategy. Although it requires extra time to wait for the convergence of the steering angle error φ_{ie}, the strategy is sufficient for performing the position movements in Eq. (55), such that the *SRC* joint can reach the required position $\boldsymbol{q}^r_i$.

Since the trolley is only required to steer first, one has $u_{if}^r = 0$. Therefore, the control rule of the trolley Eq. (57) is simplified as $u_{is} = u_{is}^r$ in Eq. (56), which guarantees that the steering angle converges to the reference direction φ_i^r.

Then, we design that the trolley moves forward with the speed

$$u_{if} = \dot{s}_i^r + k_{s_i}(s_i^r - s_i), \tag{63}$$

where k_{s_i} are positive constant gains.

Given the forward dynamics of the trolley i that $\dot{s}_i = u_{if}$, it is readily to prove that the forward speed in Eq. (63) guarantees the trolley going through the required arc distance s_i^r. Therefore, the *SRC* joint i reaches the required position $\boldsymbol{q}_i^r$. As a result, the end-effector reaches the desired pose ${}_I^I\boldsymbol{r}_{B_n}^d$ and $\boldsymbol{q}_{n,I}^d$ in the task space.

3) wheel speeds: For the FreeBOT shown in Fig. 3, the forward and steering speeds of the driving trolley and the rotation speeds of its two differential wheels satisfy the following relationships (refer to the Appendix C)

$$\begin{cases} u_{if} = \dfrac{r_{i,\text{out}}\, r_{i,w}}{\sqrt{4\, r_{i,\text{in}}^2 - l_{i,w}^2}} \left(\omega_{ir} + \omega_{il}\right), \\ u_{is} = \dfrac{r_{i,w}}{l_{i,w}} \left(\omega_{ir} - \omega_{il}\right), \end{cases} \tag{64}$$

where w_{ir}, w_{il} represents the rotation speeds of the right and left wheels of the trolley i, respectively. Other parameters are defined in the Fig. 5.

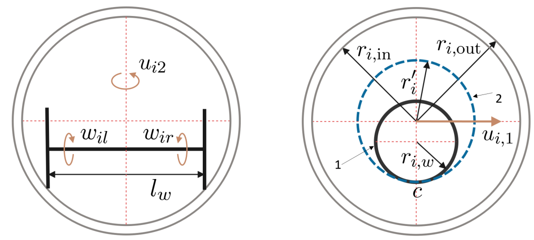
Having obtained the trolley forward and steering speeds u_{if}, u_{is} in the trolley control method, the wheel speeds that generate the u_{if}, u_{is} can be solved from Eq. (64):

$$\begin{cases} w_{ir} = \dfrac{\sqrt{4\, r_{i,\text{in}}^2 - l_{i,w}^2}}{2\, r_{i,\text{out}}\, r_{i,w}} u_{if} + \dfrac{l_{i,w}}{2\, r_{i,w}} u_{is}, \\ w_{il} = \dfrac{\sqrt{4\, r_{i,\text{in}}^2 - l_{i,w}^2}}{2\, r_{i,\text{out}}\, r_{i,w}} u_{if} - \dfrac{l_{i,w}}{2\, r_{i,w}} u_{is}. \end{cases} \tag{65}$$

Therefore, by applying rotation speeds in Eq. (65) to the differential wheels, the *SRC* joint and end-effector can realize the reference velocities, as well as that the *SRC* joint can reach the reference joint position and the end-effector reaches the desired pose. A whole flow-chart that brings the end-effector using FreeBOTs is presented in Fig. 6.

V. Experiments

This paper constructs a physics simulation system for the *SRC* joint and manipulator composed of FreeBOTs using the *Simscape Multibody* environment [33], termed as FreeBOT-SIM. The gravity, collision detection and frictions, as well as the effects of magnetic forces among the FreeBOTs can be successfully simulated. The wheel speeds of the driving trolley solved in Section IV-C are applied to the wheel block in FreeBOT-SIM, which causes the motions of the FreeBOT-SIM according to the physical laws. The FreeBOT-SIM provides a wide variety of sensors to measure the states of FreeBOTs, the *SRC* joint and manipulator. As a result, the *SRC* joint model, the kinematics models as well as the path planning and control method of FreeBOTs can be readily validated using the FreeBOT-SIM.



(a) Rear view (b) Right view

Fig. 5: Geometry parameters of the FreeBOT, $l_{i,w}$ is the wheel interval of the trolley, $r_{i,w}$ is the wheel radius of the trolley, $r_{i,\text{out}}, r_{i,\text{in}}$ are the outer and inner radii of the shell of the FreeBOT.

A physical *SRC* joint manipulator is also built using real FreeBOTs, on which the effectiveness of the manipulator kinematics models as well as the path planning and control method of FreeBOTs are validated, by realizing that the manipulator moves as expected.

A. SRC joint model validation

This section first validates the established 2-DOF *SRC* joint model. Considering that the sizes of the two bodies connected by the *SRC* joint is generally different, as shown in Fig. 7, we form the 2-DOF *SRC* joint in FreeBOT-SIM instead of using two real FreeBOTs that have the same size. The radii of the bodies B_i and B_{i-1} are 0.03m and 0.04m, respectively. Assigning the angular velocity ${}_{P_i}^{P_i}\omega_{B_{ix}} = -0.1\,\text{rad/s}$, ${}_{P_i}^{P_i}\omega_{B_{iy}} = -0.05\,\text{rad/s}$ at the *SRC* joint i, the motion sequences of the *SRC* joint at different time instants are presented in Fig. 7.

In order to validate the relationship between the *SRC* joint velocity ${}_{P_i}^{B_{i-1}}\boldsymbol{\omega}_{B_i}$ and ${}_{P_i}^{P_i}\omega_{B_{ix}}, {}_{P_i}^{P_i}\omega_{B_{iy}}$ in Eq. (1), the calculated *SRC* joint velocity using Eq. (1) and that measured using the FreeBOT-SIM sensor are presented in Fig. 8. The value consistency implies the correctness of the relationship in Eq. (1).

The rotation matrix of the *SRC* joint i at different time instants when given ${}_{P_i}^{P_i}\omega_{B_{ix}}, {}_{P_i}^{P_i}\omega_{B_{iy}}$ is calculated using the model equations in section II, which represents the orientation of the body B_i relative to the body B_{i-1}. The orientation errors between the calculated values and that measured using FreeBOT-SIM sensor are presented in Fig. 9, for intuitive purposes, which is represented using *X-Y-Z* Euler angles. It can be seen that the peak angle error is smaller than $2{\times}10^{-2°}$, and the average angle error is $[4.92, 0.01, 9.83] \times 10^{-4°}$, which is mainly caused by the numerical integration errors. Therefore, the correctness of the rotation matrix model of the *SRC* joint is validated.

When applying the *SRC* joint velocity, the generated spatial velocity of the body B_i relative to B_{i-1}, ${}_{B_{i-1}}^{B_{i-1}}\boldsymbol{\nu}_{B_i}$, are calculated through the free modes matrix $\boldsymbol{\Phi}_i$ in Eq. (8). The calculated spatial velocity ${}_{B_{i-1}}^{B_{i-1}}\boldsymbol{\nu}_{B_i}$ and that measured using the FreeBOT-SIM sensors are presented in Fig. 10. The coincidence of the calculated and measured values validates the correctness of the free modes model of the *SRC* joint.

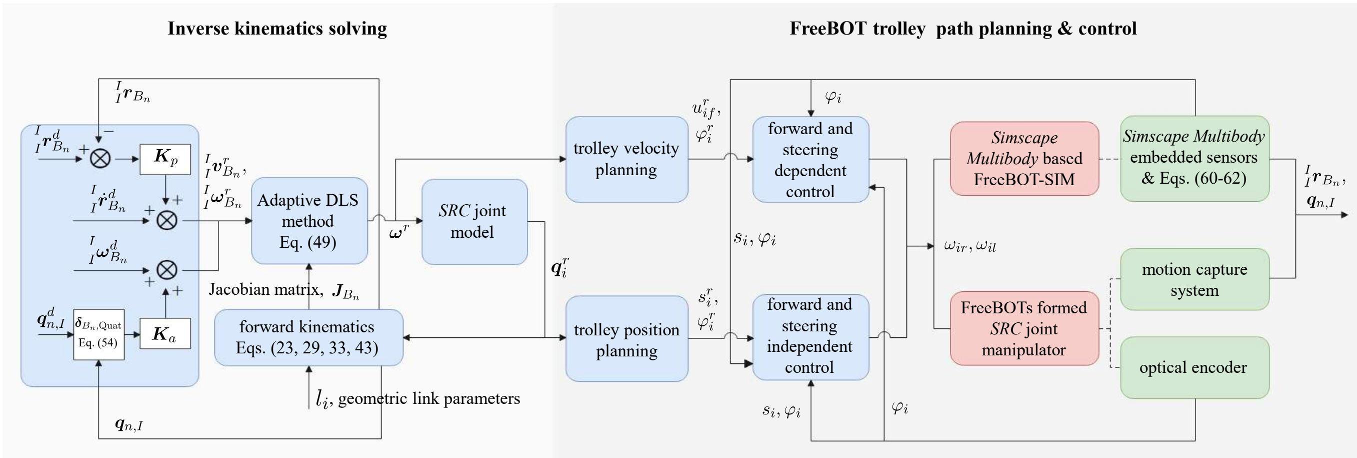
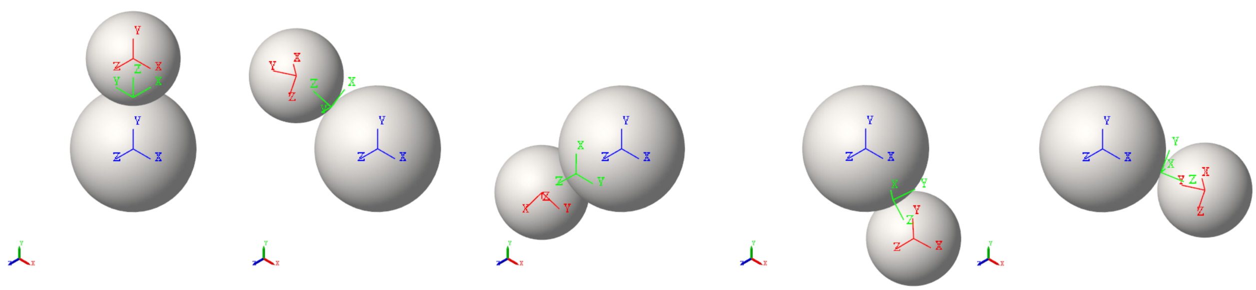


Fig. 6: Control diagram of the *SRC* joint manipulator using FreeBOTs



(a) $t = 0\,\mathrm{s}$ (b) $t = 5.9\,\mathrm{s}$ (c) $t = 14.9\,\mathrm{s}$ (d) $t = 24.2\,\mathrm{s}$ (e) $t = 30\,\mathrm{s}$

Fig. 7: 2-DOF *SRC* joint motion frames in FreeBOT-SIM with different body sizes

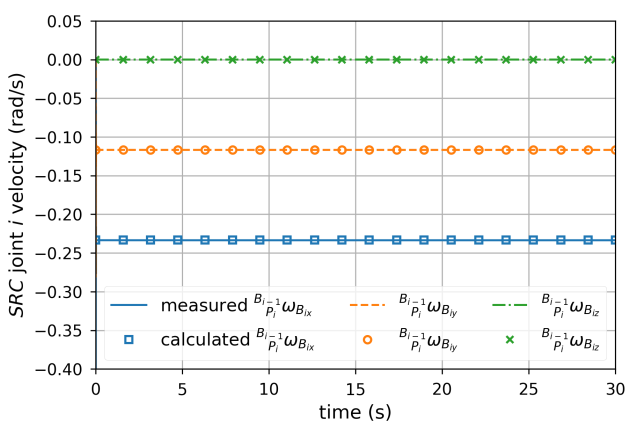


Fig. 8: 2-DOF *SRC* joint velocity measured and calculated using Eq. (1)

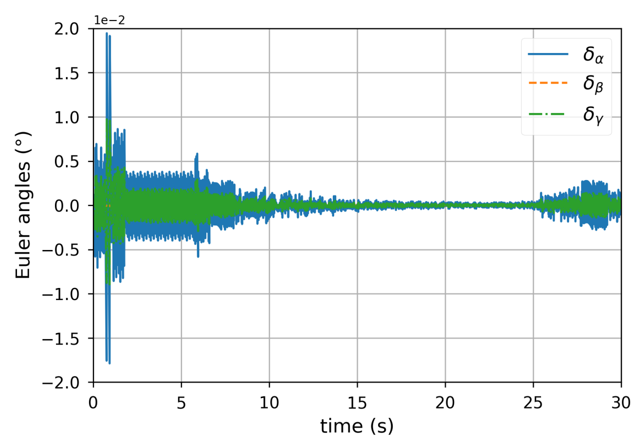


Fig. 9: Euler angles of the rotation matrix error of the *SRC* joint

B. SRC joint manipulator reaching target pose

As shown in Fig. 4, a 6-DOF *SRC* joint manipulator is built using FreeBOTs, and its simulation counterpart is constructed in the FreeBOT-SIM. In tasks where the end-effector of the *SRC* joint manipulator needs to reach a target pose, the required *SRC* joint positions are first solved using the proposed IK method in Section III-B, and then realzied with the FreeBOT driving trolley motions using the proposed path planning and control method. The kinematic geometry and control parameters of the *SRC* joint manipulator are listed in Table II.

Supposing that the end-effector needs to reach the target poses A and B, as shown in Table II, respectively. The corresponding *SRC* joint reference positions $\boldsymbol{q}_i^r$ are solved using the IK method

$$\begin{cases} \boldsymbol{q}_{1,A}^r = \begin{bmatrix} 0.7992 & 0.1912 & -0.5698 & 0.0016 \end{bmatrix}, \\ \boldsymbol{q}_{2,A}^r = \begin{bmatrix} -0.5579 & -0.1939 & -0.7714 & -0.2366 \end{bmatrix}, \\ \boldsymbol{q}_{3,A}^r = \begin{bmatrix} 0.1270 & -0.4998 & -0.3887 & 0.7636 \end{bmatrix}, \end{cases} \tag{66}$$

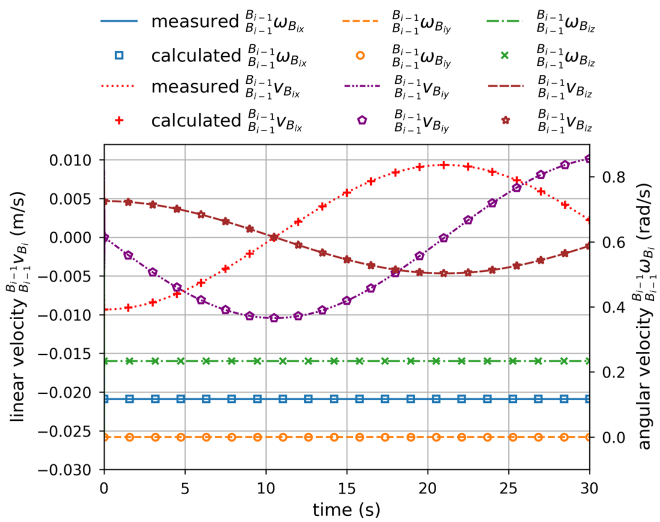


Fig. 10: Spatial velocity of B_i relative to B_{i-1} measured and calculated using free modes model

TABLE II: Kinematic geometry and control parameters of the *SRC* joint manipulator

Parameters	Values
$r_{i,\mathrm{in}}$, $r_{i,\mathrm{out}}$, l_i	0.032, 0.034, 0.034 m
$r_{i,w}$, $l_{i,w}$	0.005, 0.046 m
ξ, $\lambda_{\max}$	0.01
$\boldsymbol{K}_p$, $\boldsymbol{K}_a$	$2\boldsymbol{I}_n$
k_{s_i}, $k_{\varphi_{i,1}}$, $k_{\varphi_{i,2}}$	3, 6, 1
${}^{I}_{I}\boldsymbol{r}_{B_0}$	$\begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$ m
$\boldsymbol{q}_{B_0,I}$	$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^\top$
$\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$, $\boldsymbol{q}_4$ at t_0	$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^\top$
$\boldsymbol{\varphi}(t_0)$	$\begin{bmatrix}0.33 & 0.42 & 0.30 & 0.52\end{bmatrix}^\top$ rad
target ${}^{I}_{I}\boldsymbol{r}_A$	$\begin{bmatrix}0.0036 & 0.1636 & 0.0924\end{bmatrix}^\top$ m
target ${}^{I}_{I}\boldsymbol{r}_B$	$\begin{bmatrix}0.1151 & 0.1489 & 0.0291\end{bmatrix}^\top$ m
target ${}^{I}_{I}\boldsymbol{r}_C$	$\begin{bmatrix}0.0924 & 0.1597 & 0.1154\end{bmatrix}^\top$ m
target $\boldsymbol{q}_{A,I}$	$\begin{bmatrix}0.8610 & 0.4687 & 0.1356 & 0.1437\end{bmatrix}^\top$
target $\boldsymbol{q}_{B,I}$	$\begin{bmatrix}0.8520 & 0.3289 & 0.2300 & 0.3362\end{bmatrix}^\top$
target $\boldsymbol{q}_{C,I}$	$\begin{bmatrix}0.9682 & 0.0874 & 0.2220 & 0.0756\end{bmatrix}^\top$
obstacle 1, position	$\begin{bmatrix}0.0948 & 0.0985 & 0.0467\end{bmatrix}^\top$ m
obstacle 2, position	$\begin{bmatrix}0.0948 & 0.2107 & 0.0467\end{bmatrix}^\top$ m

and

$$\begin{cases} \boldsymbol{q}^r_{1,B} = \begin{bmatrix}-0.4558 & 0.5135 & -0.7261 & -0.0367\end{bmatrix}, \\ \boldsymbol{q}^r_{2,B} = \begin{bmatrix}-0.0130 & -0.6933 & 0.0267 & -0.7200\end{bmatrix}, \\ \boldsymbol{q}^r_{3,B} = \begin{bmatrix}0.0880 & 0.2577 & 0.8231 & -0.4983\end{bmatrix}. \end{cases} \tag{67}$$

Given the reference positions of the *SRC* joints, the trolley position planning (in Section IV-B2) and the *forward and steering independent* control (in Section IV-C2) strategy are adopted to drive the *SRC* joints to the reference joint positions. The required measurements of the real-time trolley direction φ_i and arc distance s_i are measured with simulated sensors and calculated using Eqs. (53)-(55) in the FreeBOT-SIM, and converted from the counts of the optical encoder mounted on the wheels in the physical manipulator. The pose of the end-effetor is measured using the related FreeBOT-SIM sensors and motion capture system in the FreeBOT-SIM and physical system, respectively.

Some motion frames captured at different time instants during the *SRC* joint manipulator reaching the target pose A and B are presented in Figs. 11 and 12, respectively. It can be seen that the simulated and physical manipulators go through the similar trajectories.

In order to show the motion precision of the *SRC* joint manipulator under the proposed method, the position and orientation errors of the end-effector during reaching the target poses A and B are presented in Figs. 13 and 14, respectively. It can be seen that the position and orientation errors of the end-effector finally converge to zero in both cases in the FreeBOT-SIM, which validates the effectiveness of the kinematics modeling of the *SRC* joint manipulator and the path planning and control methods of FreeBOTs. The final end-effector position errors are $[4.17, -0.21, -4.19]$mm and $[-3.51, 3.9, 1.47]$mm, and orientation errors (represented with *X-Y-Z* Euler angles) are $[1.07, -0.8, 1.45]^\circ$ and $[0.13, 1.86, 0.52]^\circ$ for the physical manipulator reaching the target pose A and B, respectively. The experimental error is mainly caused by the deformation of the anti-slip EVA foam layer, which makes the spherical shell radius not exactly the same in the experiment and simulation. A slight slip may also happen due to the deformation of the EVA foam layer along the gravity direction.

C. *SRC* joint manipulator avoiding obstacles

In a cluttered environment with obstacles, the *SRC* joint manipulator needs to complete specific tasks while avoiding obstacles. The *SRC* joint manipulator has the advantage of performing tasks in environments with obstacles or narrow space due to its geometric symmetry of spherical units. As shown in Fig. 15, the *SRC* joint manipulator needs to reach for the target pose C through the narrow space between the horizontal bars. Herein, the manipulator adopts the velocity damping method [34] to formulate the obstacle avoidance constraints using the established velocity kinematic equations (29). A sequence of quadratic programs (QP) that generalize the task-priority framework [35] are solved for the reference joint velocity, which formulate reaching the target pose and avoiding obstacles as equality and inequality constraints on joint velocity, respectively. Once the reference *SRC* joint velocity is obtained, the trolley velocity planning (in Section IV-B1) and the *forward and steering dependent* control (in Section IV-C1) strategy are used to ensure that the *SRC* joint moves along the reference velocity, so that the manipulator can avoid obstacles and the end-effector can reach the target pose.

The task-related parameters are listed in Table II. As shown in Fig. 15, the *SRC* joint manipulator smoothly avoids the obstacles in the narrow space and makes the end-effector successfully reach the target pose. It is worth pointing out that, the *SRC* joint can easily roll towards the generated obstacle avoidance direction and determine the closest point on the linkage to the obstacle, due to the geometric symmetry of the spherical shell.

The position and orientation deviations of the end-effector during the whole movement are shown in Fig. 16, and the closest distances on the manipulator to the obstacles are shown

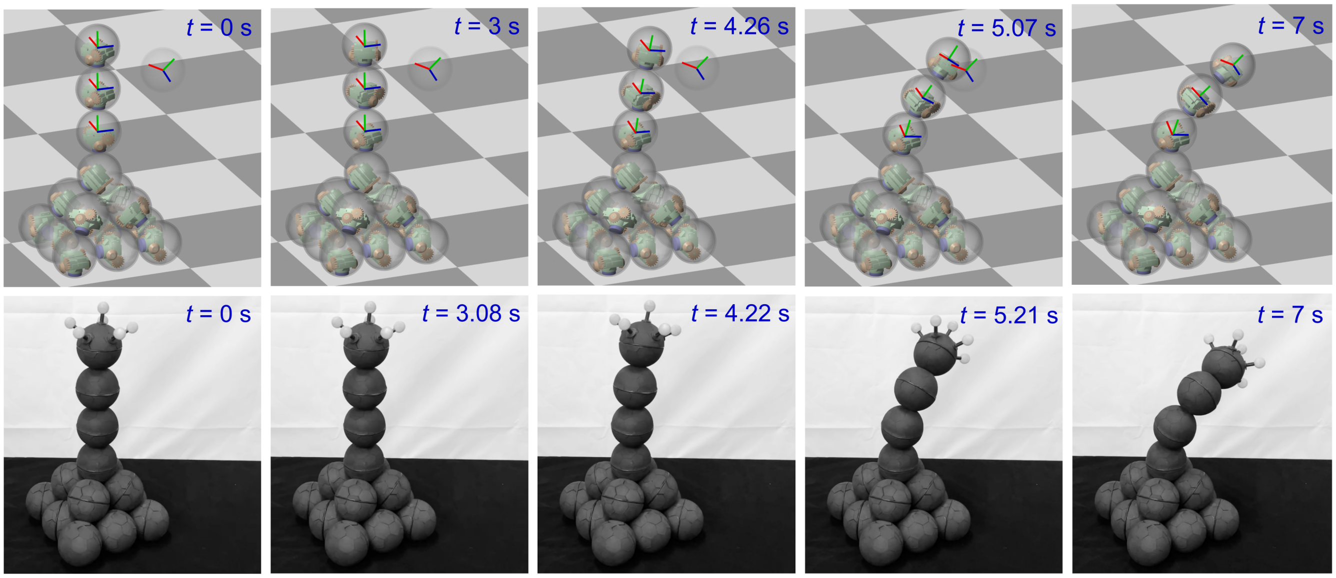


Fig. 11: Motion frames of the 6-DOF *SRC* joint manipulator reaching the target pose *A*.

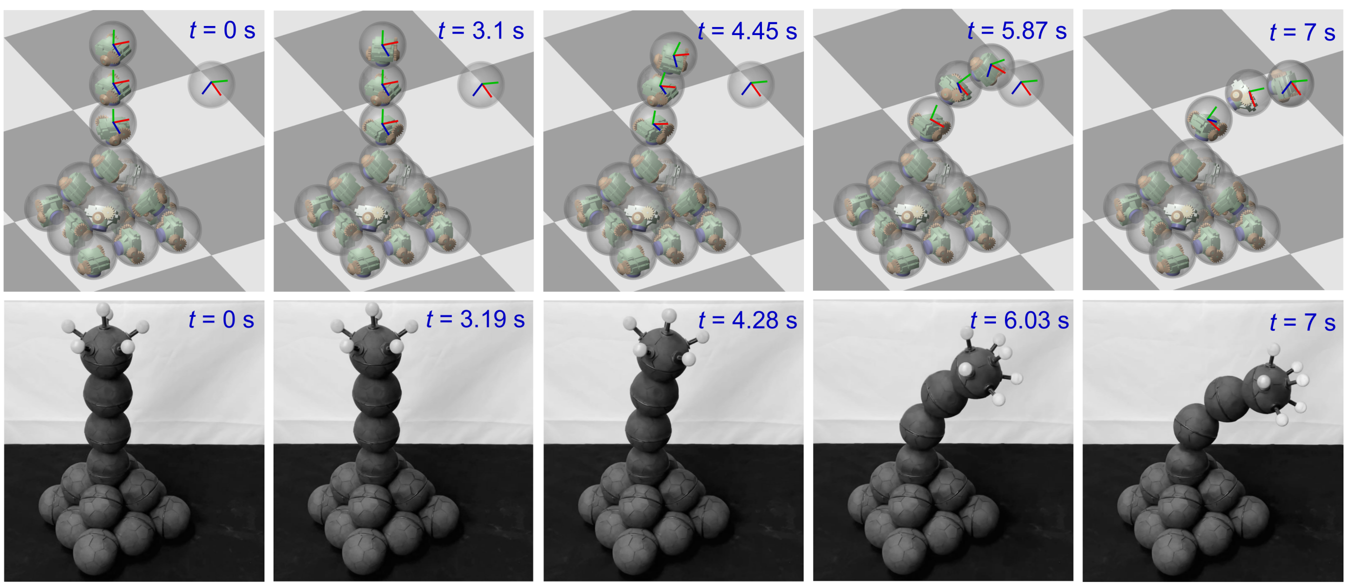
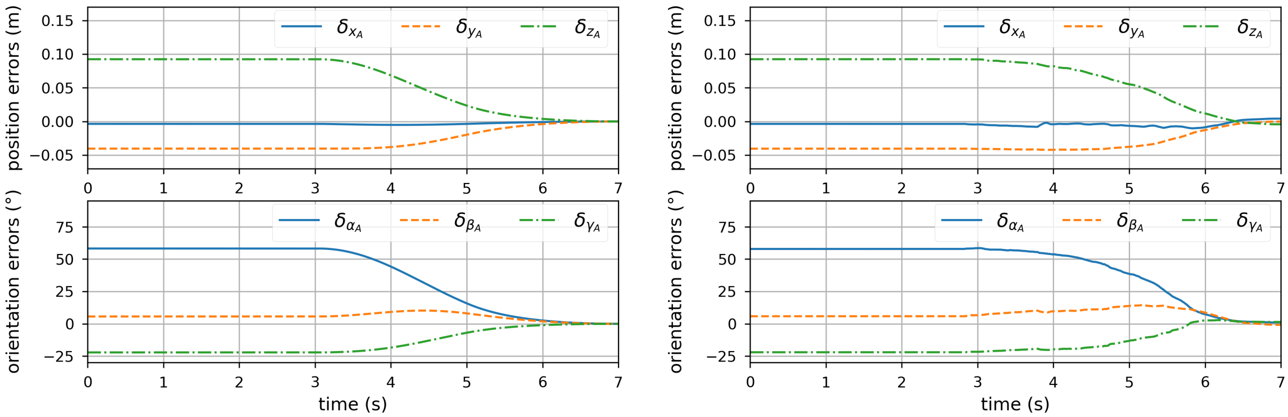


Fig. 12: Motion frames of the 6-DOF *SRC* joint manipulator reaching the target pose *B*.

in Fig. 17. It can be seen that the target pose is reached accurately, and the distances to the obstacles are always greater than the set safety distance of 0.01m, which validates the performance of the *SRC* joint manipulator in performing tasks in narrow spaces with obstacles.

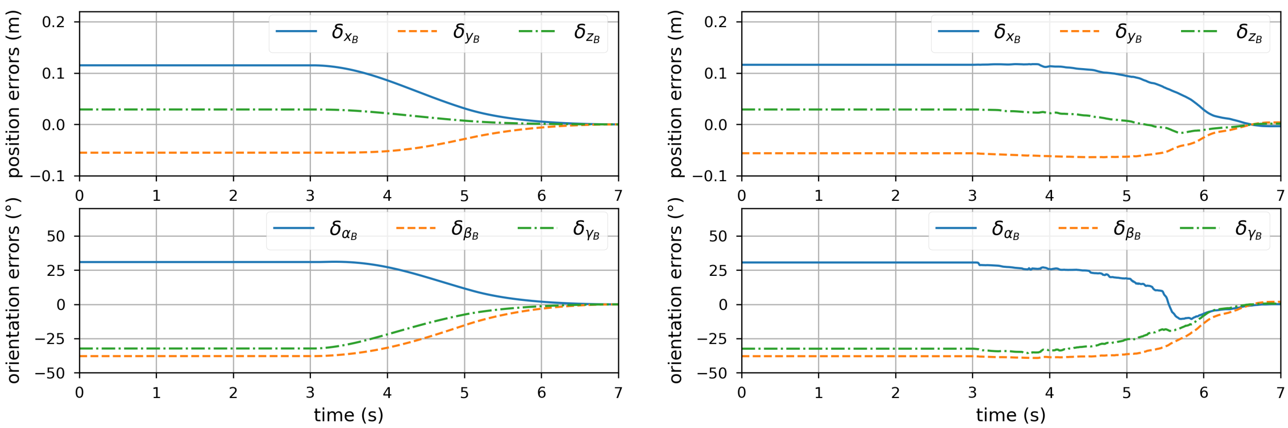
VI. Conclusions

Rolling contact joints have attracted interests in the field of robotics. This paper first proposes a novel 2-DOF *SRC* joint, with its joint model being formulated thoroughly. Then, a serial manipulator formed by the 2-DOF *SRC* joint, termed as the *SRC* joint manipulator, is presented, whose forward kinematics are established based on the 2-DOF *SRC* joint model. The Jacobian matrix that relates the *SRC* joint velocity and the end-effector velocity are obtained, and the inverse kinematics is solved using the adaptive DLS method. The FreeBOT is used to implement the motions of the 2-DOF *SRC* joint and manipulator, where the path planning and control methods, including the velocity/position planning as well as forward and steering dependent/independent control, for the driving trolley are proposed. As a result, the *SRC* joint and manipulator can generate the reference velocities and reach the target positions. In addition, a physics simulation platform FreeBOT-SIM and a real manipulator composed of FreeBOTs are built for the *SRC* joint and manipulator, on which the *SRC* joint model, the kinematics of the *SRC* joint manipulator, and the path planning and control methods of the FreeBOTs are validated, through performing *SRC* joint motions as well as the tasks of the manipulator reaching target poses in a free or obstructed space. In future applications, the *SRC* joint manipulator could work in the environments with obstacles or narrow spaces, for example, embedded pipes in buildings and truss mechanisms for some infrastructure, using the geometric symmetry advantage of its spherical units. Motion planning of the *SRC* joint manipulator performing tasks in narrow and cluttered environments with complex obstacles will be further studied in future work.



(a) simulation (b) experiment

Fig. 13: Position and orientation errors of the end-effector for reaching the target pose A



(a) simulation (b) experiment

Fig. 14: Position and orientation errors of the end-effector for reaching the target pose B

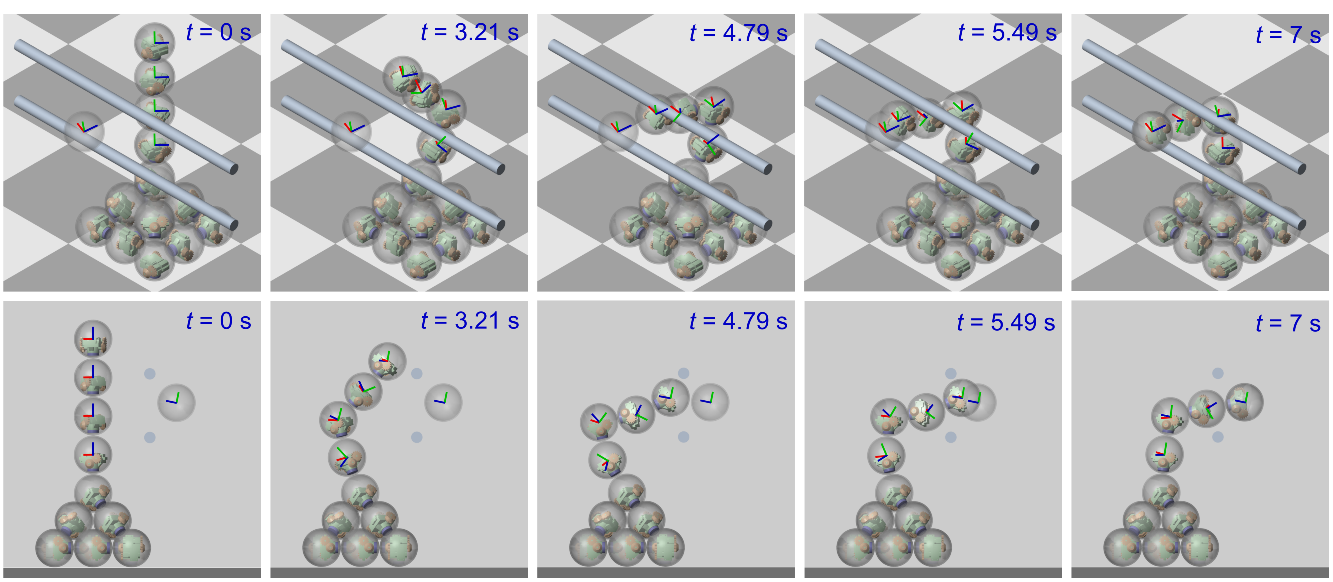


Fig. 15: Motion frames of the 8-DOF *SRC* joint manipulator reaching the target pose with obstacle avoidance.

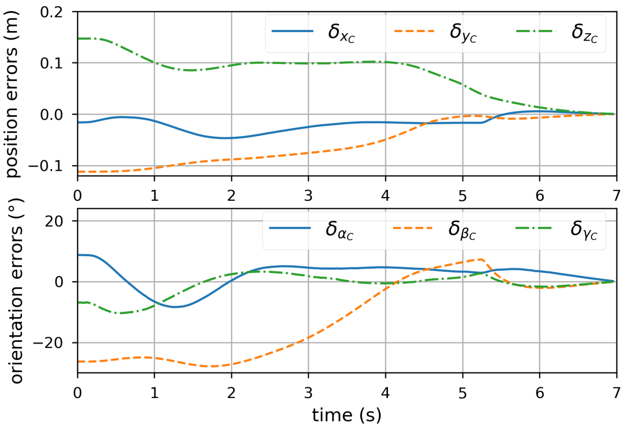


Fig. 16: Position and orientation errors of the end-effector for reaching the target pose C.

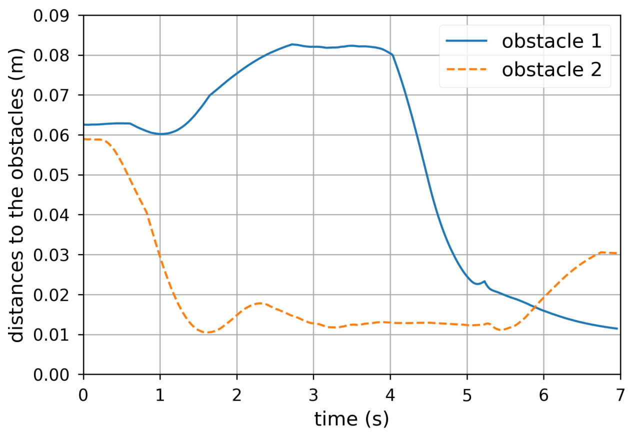


Fig. 17: Closest distances between the *SRC* joint manipulator and obstacles.

APPENDIX A
DERIVATIONS IN THE *SRC* JOINT MODELING

A. Angle-axis relationships between the bodies B_i and $B_{i\text{-}1}$ relative to the tangent plane P_i

In the inertial frame, since the body B_i and $B_{i\text{-}1}$ have the same linear velocity at the contact point, one has

$$l_i\dot{\theta}_i = l_{i\text{-}1}\dot{\theta}_{i\text{-}1}, \tag{68}$$

where $\dot{\theta}_i$ and $\dot{\theta}_{i\text{-}1}$ are rotation speeds of B_i and $B_{i\text{-}1}$ relative to the inertial frame. The rotation speed of the tangent plane P_i satisfies the relationship:

$$\dot{\theta}_{p_i} = \dot{\theta}_i - \dot{\theta}_{i\text{-}1}. \tag{69}$$

The rotation speeds of B_i and $B_{i\text{-}1}$ relative to the dynamic tangent plane P_i satisfy

$$\begin{cases}\dot{\theta}_{i,1} = \dot{\theta}_i - \dot{\theta}_{p_i},\\ \dot{\theta}_{i\text{-}1,2} = \dot{\theta}_{i\text{-}1} + \dot{\theta}_{p_i}.\end{cases} \tag{70}$$

Substituting Eq. (69) into Eq. (70) results in

$$\frac{\dot{\theta}_{i,1}}{\dot{\theta}_{i\text{-}1,2}} = \frac{\dot{\theta}_{i\text{-}1}}{\dot{\theta}_i} = \frac{l_i}{l_{i\text{-}1}} \tag{71}$$

B. Conversion between Angle-axis and quaternion representations

The angular displacement of $\Delta\boldsymbol{q}_{i,1} = [\Delta\eta_{i,1}, \Delta\boldsymbol{\epsilon}_{i,1}]^\top \in \mathbb{R}^4$ is transformed into the Angle-axis representation $\Delta\theta_{i,1}\hat{\boldsymbol{e}}_{i,1}$ using the relationship [8]

$$\begin{cases}\Delta\theta_{i,1} = 2\cos^{-1}(\Delta\eta_{i,1}),\\ \hat{\boldsymbol{e}}_{i,1} = \dfrac{\Delta\boldsymbol{\epsilon}_{i,1}}{\sin(\Delta\theta_{i,1}/2)}.\end{cases} \tag{72}$$

The unit quaternion representation of the angular displacement of $\sum B_{i\text{-}1}$ relative to $\sum P_i$, denoted as $\Delta\boldsymbol{q}_{i\text{-}1,2} = [\Delta\eta_{i\text{-}1,2}, \Delta\boldsymbol{\epsilon}_{i\text{-}1,2}]^\top \in \mathbb{R}^4$, can be calculated from $\Delta\theta_{i\text{-}1,2}\hat{\boldsymbol{e}}_{i\text{-}1,2}$ as follows

$$\begin{cases}\Delta\eta_{i\text{-}1,2} = \cos(\Delta\theta_{i\text{-}1,2}/2),\\ \Delta\boldsymbol{\epsilon}_{i\text{-}1,2} = \hat{\boldsymbol{e}}_{i\text{-}1,2}\sin(\Delta\theta_{i\text{-}1,2}/2).\end{cases} \tag{73}$$

C. Calculating rotation matrix from the corresponding unit quaternion

The rotation matrix ${}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i}$ is calculated from the unit quaternion $\boldsymbol{q}_i$ as shown in Eq. (74) [8]:

Similar to Eq. (74), the rotation matrix ${}^{B_{i\text{-}1}}\boldsymbol{R}_{P_i}$ can be expressed as the functions of the unit quaternion $\boldsymbol{q}_{i\text{-}1,2}$ in Eq. (10), where

$$\boldsymbol{f}_1(\boldsymbol{q}_{i\text{-}1,2}) = \begin{bmatrix}1-2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(2)^2+\boldsymbol{\epsilon}_{i\text{-}1,2}(3)^2\right)\\ 2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(1)\boldsymbol{\epsilon}_{i\text{-}1,2}(2)-\eta_{i\text{-}1,2}\boldsymbol{\epsilon}_{i\text{-}1,2}(3)\right)\\ 2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(1)\boldsymbol{\epsilon}_{i\text{-}1,2}(3)+\eta_{i\text{-}1,2}\boldsymbol{\epsilon}_{i\text{-}1,2}(2)\right)\end{bmatrix}, \tag{75}$$

$$\boldsymbol{f}_2(\boldsymbol{q}_{i\text{-}1,2}) = \begin{bmatrix}2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(1)\boldsymbol{\epsilon}_{i\text{-}1,2}(2)+\eta_{i\text{-}1}\boldsymbol{\epsilon}_{i\text{-}1,2}(3)\right)\\ 1-2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(1)^2+\boldsymbol{\epsilon}_{i\text{-}1,2}(3)^2\right)\\ 2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(2)\boldsymbol{\epsilon}_{i\text{-}1,2}(3)-\eta_{i\text{-}1,2}\boldsymbol{\epsilon}_{i\text{-}1,2}(1)\right)\end{bmatrix}, \tag{76}$$

$$\boldsymbol{f}_3(\boldsymbol{q}_{i\text{-}1,2}) = \begin{bmatrix}2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(1)\boldsymbol{\epsilon}_{i\text{-}1,2}(3)-\eta_{i\text{-}1,2}\boldsymbol{\epsilon}_{i\text{-}1,2}(2)\right)\\ 2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(2)\boldsymbol{\epsilon}_{i\text{-}1,2}(3)+\eta_{i\text{-}1,2}\boldsymbol{\epsilon}_{i\text{-}1,2}(1)\right)\\ 1-2\left(\boldsymbol{\epsilon}_{i\text{-}1,2}(1)^2+\boldsymbol{\epsilon}_{i\text{-}1,2}(2)^2\right)\end{bmatrix}. \tag{77}$$

D. Calculating linear velocity ${}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i}$

$$\begin{aligned}{}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{v}_{B_i} &= {}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{P_i} \times \left({}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{r}_{B_i}\right)\\ &= -\left({}^{B_{i\text{-}1}}\boldsymbol{R}_{P_i}\,{}^{B_{i\text{-}1}}_{P_i}\boldsymbol{r}_{B_i}\right) \times \left({}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{P_i}\right)\\ &= -\left({}^{B_{i\text{-}1}}\boldsymbol{R}_{P_i}\,[0,0,l_i+l_{i\text{-}1}]^\top\right) \times \left({}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{P_i}\right)\\ &= -(l_i+l_{i\text{-}1})\boldsymbol{f}_3^\times(\boldsymbol{q}_{i\text{-}1,2})\left({}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{P_i}\right)\\ &= (l_i+l_{i\text{-}1})\boldsymbol{f}_3^\times(\boldsymbol{q}_{i\text{-}1,2})\left({}^{P_i}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_{i\text{-}1}}\right).\end{aligned} \tag{78}$$

$${}^{B_{i\text{-}1}}\boldsymbol{R}_{B_i} = \begin{bmatrix}1-2\left(\boldsymbol{\epsilon}_i(2)^2+\boldsymbol{\epsilon}_i(3)^2\right) & 2\left(\boldsymbol{\epsilon}_i(1)\boldsymbol{\epsilon}_i(2)-\eta_i\boldsymbol{\epsilon}_i(3)\right) & 2\left(\boldsymbol{\epsilon}_i(1)\boldsymbol{\epsilon}_i(3)+\eta_i\boldsymbol{\epsilon}_i(2)\right)\\ 2\left(\boldsymbol{\epsilon}_i(1)\boldsymbol{\epsilon}_i(2)+\eta_i\boldsymbol{\epsilon}_i(3)\right) & 1-2\left(\boldsymbol{\epsilon}_i(1)^2+\boldsymbol{\epsilon}_i(3)^2\right) & 2\left(\boldsymbol{\epsilon}_i(2)\boldsymbol{\epsilon}_i(3)-\eta_i\boldsymbol{\epsilon}_i(1)\right)\\ 2\left(\boldsymbol{\epsilon}_i(1)\boldsymbol{\epsilon}_i(3)-\eta_i\boldsymbol{\epsilon}_i(2)\right) & 2\left(\boldsymbol{\epsilon}_i(2)\boldsymbol{\epsilon}_i(3)+\eta_i\boldsymbol{\epsilon}_i(1)\right) & 1-2\left(\boldsymbol{\epsilon}_i(1)^2+\boldsymbol{\epsilon}_i(2)^2\right)\end{bmatrix}. \tag{74}$$

APPENDIX B
DERIVATIONS IN THE MANIPULATOR MODELING

A. Iterative process of calculating ${}^{I}_{I}\boldsymbol{\omega}_{B_i}$

$$\begin{aligned}
{}^{I}_{I}\boldsymbol{\omega}_{B_i} &= {}^{I}\boldsymbol{R}_{B_{i\text{-}1}}\,{}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i} + {}^{I}_{I}\boldsymbol{\omega}_{B_{i\text{-}1}} \\
&= {}^{I}\boldsymbol{R}_{B_{i\text{-}1}}\,{}^{B_{i\text{-}1}}_{B_{i\text{-}1}}\boldsymbol{\omega}_{B_i} + {}^{I}\boldsymbol{R}_{B_{i\text{-}2}}\,{}^{B_{i\text{-}2}}_{B_{i\text{-}2}}\boldsymbol{\omega}_{B_{i\text{-}1}} + {}^{I}_{I}\boldsymbol{\omega}_{B_{i\text{-}2}} \\
&= \sum_{j=1}^{i}\left({}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\,{}^{B_{j\text{-}1}}_{B_{j\text{-}1}}\boldsymbol{\omega}_{B_j}\right) + {}^{I}_{I}\boldsymbol{\omega}_{B_0} \\
&= \sum_{j=1}^{i}\left({}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\boldsymbol{\Phi}_{j\boldsymbol{\omega}}\begin{bmatrix} {}^{B_{j\text{-}1}}_{P_j}\omega_{B_{jx}} \\ {}^{B_{j\text{-}1}}_{P_j}\omega_{B_{jy}} \end{bmatrix}\right) + {}^{I}_{I}\boldsymbol{\omega}_{B_0}.
\end{aligned} \tag{79}$$

B. Calculating linear velocity ${}^{I}_{I}\boldsymbol{v}_{B_i}$

Differentiating Eq. (26) results in

$$\begin{aligned}
{}^{I}_{I}\boldsymbol{v}_{B_i} =& {}^{I}_{I}\boldsymbol{v}_{B_0} + \sum_{j=1}^{i}\left({}^{I}_{I}\boldsymbol{\omega}^{\times}_{B_{j\text{-}1}}\,{}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\,{}^{B_{j\text{-}1}}_{B_{j\text{-}1}}\boldsymbol{r}_{B_j}\right) + \\
& \sum_{j=1}^{i}\left({}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\,{}^{B_{j\text{-}1}}_{B_{j\text{-}1}}\boldsymbol{v}_{B_j}\right) \\
=& {}^{I}_{I}\boldsymbol{v}_{B_0} - \sum_{j=1}^{i}\left({}^{B_{j\text{-}1}}_{I}\boldsymbol{r}^{\times}_{B_j}\,{}^{I}_{I}\boldsymbol{\omega}_{B_{j\text{-}1}}\right) + \\
& \sum_{j=1}^{i}\left({}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\,{}^{B_{j\text{-}1}}_{B_{j\text{-}1}}\boldsymbol{v}_{B_j}\right).
\end{aligned} \tag{80}$$

Substituting Eqs. (22) and (28) into Eq. (80) results in

$$\begin{aligned}
{}^{I}_{I}\boldsymbol{v}_{B_i} =& {}^{I}_{I}\boldsymbol{v}_{B_0} - \sum_{j=1}^{i}\left({}^{B_{j\text{-}1}}_{I}\boldsymbol{r}^{\times}_{B_j}\left({}^{I}_{I}\boldsymbol{\omega}_{B_0} + \boldsymbol{J}_{B_{j\text{-}1}\boldsymbol{\omega}}\,\boldsymbol{\omega}\right)\right) + \\
& \sum_{j=1}^{i}\left({}^{I}\boldsymbol{R}_{B_{j\text{-}1}}\boldsymbol{\Phi}_{j\boldsymbol{v}}\begin{bmatrix} {}^{B_{j\text{-}1}}_{P_j}\omega_{B_{jx}} \\ {}^{B_{j\text{-}1}}_{P_j}\omega_{B_{jy}} \end{bmatrix}\right) \\
=& {}^{I}_{I}\boldsymbol{v}_{B_0} - \sum_{j=1}^{i}\left({}^{B_{j\text{-}1}}_{I}\boldsymbol{r}^{\times}_{B_j}\right){}^{I}_{I}\boldsymbol{\omega}_{B_0} - \\
& \sum_{j=1}^{i}\left({}^{B_{j\text{-}1}}_{I}\boldsymbol{r}^{\times}_{B_j}\boldsymbol{J}_{B_{j\text{-}1}\boldsymbol{\omega}}\right)\boldsymbol{\omega} + \boldsymbol{J}'_{B_i\boldsymbol{v}}\boldsymbol{\omega} \\
=& {}^{I}_{I}\boldsymbol{v}_{B_0} + {}^{I}_{I}\boldsymbol{r}^{\times}_{0i}\,{}^{I}_{I}\boldsymbol{\omega}_{B_0} + \boldsymbol{J}_{B_i\boldsymbol{v}}\boldsymbol{\omega},
\end{aligned} \tag{81}$$

where the expressions of ${}^{I}_{I}\boldsymbol{r}_{0i}, \boldsymbol{J}'_{B_i\boldsymbol{v}}, \boldsymbol{J}_{B_i\boldsymbol{v}}$ are shown in Eqs. (30)-(32).

APPENDIX C
RELATIONSHIPS BETWEEN THE TROLLEY FORWARD AND STEERING SPEEDS AND WHEEL SPEEDS

As shown in Fig. 5b [8], wheel 1 and virtual wheel 2 have the same velocity at the tangent point c:

$$r_{iw}\frac{1}{2}(\omega_{ir} + \omega_{il}) = r'_i\omega'_{ir}, \tag{82}$$

with

$$r'_i = \sqrt{r_{in}^2 - (l_w/2)^2}. \tag{83}$$

Therefore, one has

$$\omega'_{ir} = r_{iw}\frac{1}{2}(\omega_{ir} + \omega_{il})/\sqrt{r_{in}^2 - (l_w/2)^2}, \tag{84}$$

and

$$u_{i,1} = r_{i,\text{out}}\omega'_{ir} = \frac{r_{i,\text{out}}r_{i,w}}{\sqrt{r_{in}^2 - (l_w/2)^2}}\frac{1}{2}(\omega_{ir} + \omega_{il}). \tag{85}$$

For the rotation of the trolley, as shown in Fig. 5a, one has

$$u_{is}\frac{l_w}{2} = \frac{\omega_{ir} + \omega_{il}}{2}r_{i,w}, \tag{86}$$

which gives

$$u_{is} = \frac{r_{i,w}}{l_w}(\omega_{ir} + \omega_{il}). \tag{87}$$

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